3D gravity with propagating torsion: Hamiltonian structure of the scalar sector[1]

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Contents

[1]**Based on a joint work with B. Cvetković, arXiv:1309.0411 [gr-qc]**

1. Introduction

— **3D gravity** is a **technically simplified** model of the realistic 4D theory, possessing the **same conceptual features** (local symmetries, black holes)

— Work on **Riemannian** 3D gravity was initiated by Staruszkiewicz in **1963**.
Further developments in this field led to a number of outstanding results, such as: asymptotic symmetry with central charges (1986), CS formulation (1986), BTZ black holes (1992), AdS/CFT, etc. For a review, see Carlip (2005).

— In **1991**, Mielke and Baekler (MB) proposed a new, **non-Riemannian approach** to 3D gravity, based on **Poincaré gauge theory** (PGT), with **Riemann–Cartan** geometry of spacetime. Dynamical features of the MB model provide a strong support to the concept of **torsion** as a natural geometric ingredient of gravitational interactions.

For a review, see M.B. and Hehl (2013).

— Both **GR** and **MB model** are **topological** theories, without propagating modes. Such a situation is certainly not quite realistic. In PGT, the existence of propagating modes can be achieved simply by using Lagrangians **quadratic in the field strengths**, the torsion and the curvature. Investigation of 3D gravity **with propagating torsion** started quite recently, see M.B. and B. Cvetković **(2013)**, and references therein.

Problem

The general (parity-invariant) PGT Lagrangian is defined by 8 free parameters.
How can we find out which values of parameters are dynamically allowed?

The **weak-field approximation** (unitarity, the absence of ghosts and tachions) is not reliable, as it does not always lead to a correct description of nonlinear theories.

Following Nester et al. (1999), we use the **constrained Hamiltonian method** to identify acceptable sets of parameters, by studying the following, fully **nonlinear** dynamical effects:

- the **phenomenon of "constraint bifurcation"** (the change of the number and/or type of constraints in certain regions of spacetime);
- the **stability of the canonical structure under linearization**.

Conventions

Let $\mathcal{M}$ be a spacetime with local coordinates $x^\mu = (x^0, x^\alpha)$, and $h_n = h_n{}^\mu \partial_\mu$ a Lorentz basis in it. Then, if Σ is a spacelike hypersurface with a unit normal n_k, each vector V_k in $\mathcal{M}$ can be decomposed in terms of its normal and parallel component as follows:

$$V_k = n_k V_\perp + V_{\bar{k}}\,, \quad \text{where} \quad V_\perp := n^m V_m\,, \quad V_{\bar{k}} := h_k{}^\alpha V_\alpha\,.$$

Note that $V_{\bar{k}}$ does not contain the time component of V_μ.

2. Quadratic PGT and its scalar modes

General parity-invariant gravitational Lagrangian:

$$\begin{aligned} L_G &= -a\varepsilon_{ijk}b^i R^{jk} - \frac{1}{3}\Lambda_0\varepsilon_{ijk}b^i b^j b^k + L_{T^2} + L_{R^2}\,, \\ L_{T^2} &= T^{i\star}\left(a_1{}^{(1)}T_i + a_2{}^{(2)}T_i + a_3{}^{(3)}T_i\right)\,, \\ L_{R^2} &= \frac{1}{2}R^{ij\star}\left(b_4{}^{(4)}R_{ij} + b_5{}^{(5)}R_{ij} + b_6{}^{(6)}R_{ij}\right)\,, \end{aligned} \tag{1}$$

where ${}^{(n)}T$ and ${}^{(n)}R$ are irreducible components of torsion and curvature.

Because of the complexity of the general Hamiltonian analysis, we restrict our attention to the sectors with **scalar modes**, $J^P = 0^+$ or 0^-. These modes, defined in the weak-field approximation around **Minkowski** background, satisfy the Klein–Gordon eqs. with masses:

$$m^2_{0^+} = \frac{3a(a+a_2)}{a_2(b_4+2b_6)}\,, \qquad m^2_{0^-} = \frac{3(a-a_1)(a+2a_3)}{(a_1+2a_3)b_5}\,. \tag{2}$$

$a_2(b_4+2b_6) \neq 0 \quad \Rightarrow \quad m^2_{0^+}$ is **finite** $\quad \Rightarrow \quad$ spin-0^+ mode **propagates**, etc.

The generalization to **(A)dS** background is straightforward.

3. Primary if-constraints

The canonical momenta $\pi_i{}^\mu$ and $\Pi_{ij}{}^\mu$, corresponding to the basic dynamical variables $b^i{}_\mu$ and $A^{ij}{}_\mu$, are defined in the standard manner. Frome these definitions, one immediately obtains **10 "sure" (always present) primary constraints, $\boldsymbol{\pi_i{}^0 \approx 0}$ and $\boldsymbol{\Pi_{ij}{}^0 \approx 0}$**.

What happens with the remaining momenta $\pi_i{}^\alpha$ and $\Pi_{ij}{}^\alpha$? The irreducible decomposition of the equations defining $\hat\pi_{i\bar k} := \pi_i{}^\alpha b_{k\alpha}$ yields:

$$\phi_{\perp\bar k} \equiv \frac{\hat\pi_{\perp\bar k}}{J} - (a_2 - a_1)T^{\bar m}{}_{\bar m\bar k} = (a_1 + a_2)T_{\perp\perp\bar k}\,, \tag{3a}$$

$$^S\phi \equiv \frac{^S\hat\pi}{J} = -2a_2 T^{\bar m}{}_{\bar m\perp}\,, \tag{3b}$$

$$^A\phi_{\bar\imath\bar k} \equiv \frac{^A\hat\pi_{\bar\imath\bar k}}{J} - \frac{2}{3}(a_1 - a_3)T_{\perp\bar\imath\bar k} = -\frac{2}{3}(a_1 + 2a_3)T_{[\bar\imath\bar k]\perp}\,, \tag{3c}$$

$$^T\phi_{\bar\imath\bar k} \equiv \frac{^T\hat\pi_{\bar\imath\bar k}}{J} = -2a_1{}^T T_{\bar\imath\bar k\perp}\,, \tag{3d}$$

Thus, for instance, if $a_1 + a_2 = 0$, the expression $\phi_{\perp\bar k}$ becomes a primary constraint, etc. Such additional primary constraints are known as **primary if-constraints**.

Similar analysis can be done for $\Pi_{ij}{}^{\alpha}$, and the complete results are given in Table 1.

Table 1. Primary if-constraints

Critical conditions	Primary constraints	J^P
$\boldsymbol{a_2} = 0$ $b_4 + 2\boldsymbol{b_6} = 0$	${}^S\phi \approx 0$ ${}^S\Phi_\perp \approx 0$	0^+
$a_1 + 2a_3 = 0$ $b_5 = 0$	${}^A\phi_{\bar\imath\bar k} \approx 0$ ${}^A\Phi_{\perp\bar\imath\bar k} \approx 0$	0^-
$a_1 + \boldsymbol{a_2} = 0$ $b_4 + b_5 = 0$	$\phi_{\perp\bar k} \approx 0$ ${}^V\Phi_{\bar k} \approx 0$	1
$a_1 = 0$ $b_4 = 0$	${}^T\phi_{\bar\imath\bar k} \approx 0$ ${}^T\Phi_{\perp\bar\imath\bar k} \approx 0$	2

This classification has a highly interesting interpretation.
Example: **for $\boldsymbol{a_2(b_4 + 2b_6) \neq 0}$**, the pair of if-constraints ${}^S\phi$ and ${}^S\Phi_\perp$ is absent, and **spin-0^+ mode becomes a propagating degree of freedom**.

Analogous conclusions hold for all the other J^P sectors.

4. Spin-0^+ sector

To ensure finite mass of the spin-0^+ mode, we adopt the set of simpler conditions:

$$a_2, b_6 \neq 0\,, \qquad a_1 = a_2 = b_4 = b_5 = 0\,. \tag{4a}$$

The corresponding Lagrangian takes the form (with $V_k = T^m{}_{mk}$)

$$\mathcal{L}_G^+ = -aR - 2\Lambda_0 + \frac{1}{2}a_2 V^k V_k + \frac{1}{12}b_6 R^2\,, \tag{4b}$$

The set of all constraints is given in the following table.

Table 2. **Generic** constraints in the 0^+ sector

	First class	Second class
Primary	$\pi_i{}^0$, $\Pi_{ij}{}^0$	$\boldsymbol{{}^V\Phi_{\bar\imath}}$; ${}^A\phi$, ${}^A\Phi$, ${}^T\phi$, ${}^T\Phi$
Secondary	$\mathcal{H}'_\perp$, $\mathcal{H}'_\alpha$, $\mathcal{H}'_{ij}$	$\boldsymbol{\chi_{\bar\imath}}$

With $N = 18$, $N_1 = 12$, $N_2 = 10$, we have $N^* = 2$, which is just **1 Lagrangian DoF**, the spin-0^+ mode.

Constraint bifurcation

The determinant of the SC constraints has the form:

$$\Delta^+ = W^{10}\,(W-a_2)^4 \quad \text{where} \quad W := \frac{{}^S\Pi_\perp}{4J}\,. \tag{5}$$

Being a field-dependent object, Δ^+ may vanish in the regions where $W(W-a_2)=0$, changing thereby the canonical nature of constraints $\Rightarrow$ **constraint bifurcation!**.

By analyzing the field equation

$$-(W-a_2)V_k + 2\partial_k(W-a_2) \approx 0\,, \tag{6}$$

and assuming that W is an analytic function on $\mathcal{M}$, we concluded the following:

- **If there is a point in $\mathcal{M}$ at which $W-a_2\neq 0$, then $W-a_2\neq 0$ globally**.

Thus, by choosing the initial data so that $W-a_2\neq 0$ at $x^0=0$, it follows that $W-a_2$ stays nonvanishing for any $t>0$. Moreover, for $\boldsymbol{a_2>0}$, we have:

- **By choosing $\Omega = W-a_2>0$ at $x^0=0$, it follows that $W\neq 0$ globally**.

Thus, by a suitable choice of the initial conditions, one can avoid the constraint bifurcation problem, and the constraint structure remains the same as in Table 2.

Canonical stability under linearization

Table 3 defines characteristic sectors of the linear regime, with $W = a + a_2 + \boldsymbol{qb_6}$.

Table 3. Canonical stability in the 0^+ sector

	$a + \boldsymbol{qb_6}$	$a + a_2 + \boldsymbol{qb_6}$	DoF	stability
(a)	$\neq 0$	$\neq 0$	1	stable
(b)	$= 0$	$\neq 0$	0	unstable
(c)	$\neq 0$	$= 0$	1	stable*

(a) The nature of constraints remains the same as in Table 3, and we have a single Lagrangian DoF, the massive spin-0^+ mode [around **(A)dS** background].

(b) All if-constraints become FC, but only 6 of them are independent $\Rightarrow$ $N^* = 0$.

(c) The massless nonlinear theory, defined by the condition $a + a_2 + \mathbf{qb_6} = 0$, is essentially (up to a gauge fixing) stable under the linearization.

Concluding remarks

- By investigating the fully nonlinear "constraint bifurcation" effects, as well as the canonical stability under linearization, we were able to identify the set of dynamically acceptable values of parameters for the spin-0^+ sector, as shown in Table 3.

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- **References**

M. Blagojević and B. Cvetković, 3D gravity with propagating torsion: Hamiltonian structure of the scalar sector, e-print arXiv:1309.0411 [gr-qc].
Here, one can find all the references mentioned in this talk.

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