Three tales on boundary action in AdS/CFT correspondence

Berry phases, gauge fields and non-canonical Hamiltonian systems

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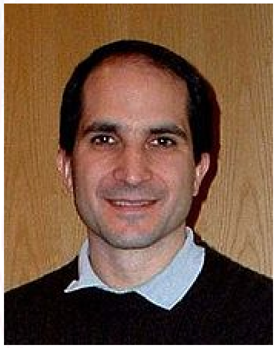
AdS/CFT in a nutshell



J. Maldacena

S. Gubser

I. Klebanov

E. Witten

- Second superstring revolution: a web of dualities (Maldacena 1997, Gubser, Klebanov & Polyakov 1998, Witten 1998)
- Stack of N parallel D3 branes: open vs. closed string description
- Open strings: N=4 U(N) super Yang-Mills
- Closed strings: type IIB supergravity on anti-de Sitter space

$$\mathrm{AdS}_{D+1} \sim \mathrm{CFT}_{D}$$

gravity in D+1 dimensions = field theory in D dimensions

Top-down and bottom-up

Top-down

- Original 1997 papers
- Start from consistent string theory action
- Know the Lagrangian and degrees of freedom on both sides
- Parameters generally fixed
- Harder to modify, add matter fields, play around

Bottom-up

- Most applied holography (cond-mat, QCD, fluids)
- Phenomenological theory in AdS, no string action
- In general don't know the field theory Lagrangian
- Free parameters

Anti-de-Sitter space: a reminder

- AdS$_{D+1}$ – maximally symmetric with constant curvature <0
- Immerse as a hyperboloid in $R^{D,2}$: $X_0^2+X_{D+1}^2-X_i X^i=L^2$
- Isommetry group: $SO(2,D)$ ~ conformal group in $R^{D-1,1}$
- Global coordinates:

$$ds^2=-\frac{dt^2}{\cos^2\theta}+\frac{d\theta^2}{\cos^2\theta}+\tan^2\theta\, d\Omega^{2,}_{D-1} \quad 0\le\theta\le\pi/2$$

- Poincare patch (blow-up around the boundary):

$$ds^2=\frac{-dt^2+dx_i dx^i+dz^2}{z^2}, \quad 0\le z\le\infty$$

Holographic dictionary I

- GKPW prescription for correlation functions (Gubser, Klebanov, Polyakov, Witten):

  $$\left\langle e^{\oint d^D x \varphi_0(x) O(x)} \right\rangle_{CFT} = Z_{AdS}\left(\varphi(x; z=0) = \varphi_0(x)\right)$$

  Generating function for the correlation functions in CFT is the partition function in AdS with appropriate bonudary conditions

  $\varphi(x, z)$ - bulk field; $O(x)$ - CFT field

- On-shell action $\Leftrightarrow$ free energy: $e^{-\beta F} = \left\langle e^{-S_{bulk} - S_{bnd}} \right\rangle_{AdS}$

Boundary values of fields in AdS = sources of fields in CFT

Outline

- *The idea:*

  Bottom-up construction leaves the surface term of the AdS action non-unique. What does it mean?

- Dirac fermion in global AdS and the holographic Berry phase
- Phase diagram – strongly coupled Dirac cones and beyond

- Non-canonical Hamiltonian systems
- Dynamical gauge fields in bottom-up setups

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Action principle in AdS

- Example: scalar in AdS

$$S=\int d^{D+1}x\sqrt{g}\left(R-\Lambda-\nabla\Phi^2-m^2\Phi^2\right)+S_{bnd}$$

- Boundary term S_{bnd}:

  - regularize the bulk action

  - ensure that total variation vanishes on-shell: $\delta S=0$

- $S_{bnd}=\oint d^{Dx}\sqrt{h}\left(K-\lambda\right)+S^{\Phi}_{bnd}$ - Hawking-Gibbons term for gravity

- Variation of the scalar term:

$$\delta S(\Phi)=\int d^{D+1}x\sqrt{g}\,\underbrace{\Phi\left(\nabla^2+m^2\right)\Phi}_{=0}+\oint d^{Dx}\sqrt{h}\frac{z}{2}\Phi\nabla\Phi+\delta S^{\Phi}_{bnd}$$

- Boundary term: $S^{\Phi}_{bnd}=-\oint d^{Dx}\sqrt{h}\frac{z}{2}\Phi\nabla\Phi$

- Cancels out the variation from the bulk

Action freedom in AdS

- ***I – change boundary conditions***

$$\Phi(z \to 0) = \varphi_+ z^{\alpha} + \varphi_- z^{D-\alpha} + \ldots, \qquad \varphi_+ = \varphi_{0,} \ \ \varphi_- = 0$$

- Double-trace deformation in QFT (Witten 2001):

$$S \to S + \oint d^D x \sqrt{h} \kappa \varphi_+ \hat{K} \varphi_- \ \Rightarrow \ \varphi_+ = \varphi_{0,} \ \ \varphi_+ + \kappa \varphi_- = 0$$

- ***II – add terms with zero variation***

$$\varphi_+ = \varphi_{0,} \ \ \varphi_- = 0, \quad S \to S + \oint d^D x \sqrt{h} F(\varphi_-) \ \Rightarrow \ \delta F = 0$$

- Change S without changing δS
- ***III – add boundary DOF decoupled from the bulk***
- Singleton representation of AdS group (Flato&Fronsdal, 80s)

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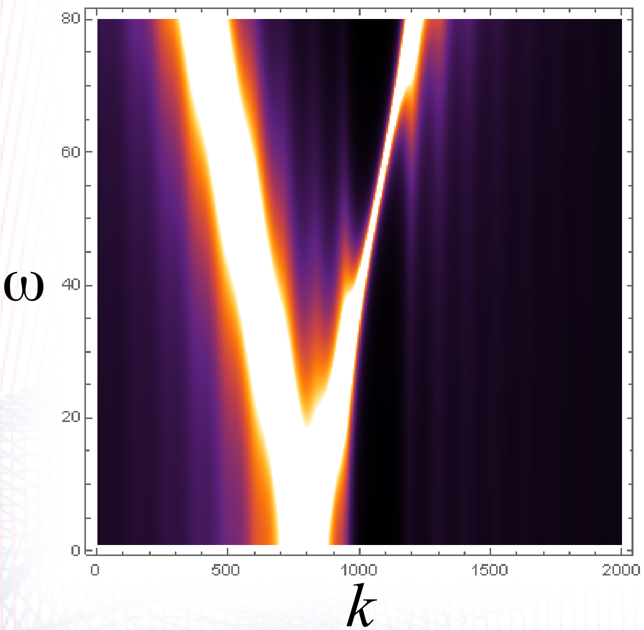
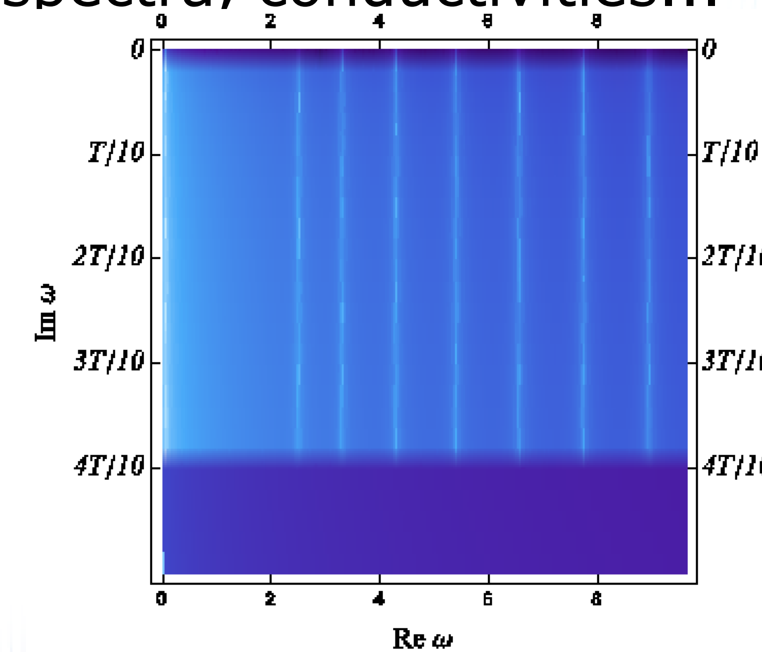
Holography for fermions

- Top-down: confinement in AdS/QCD
- Bottom-up: Dirac fermion on Poincare patch → strongly coupled fermionic system
- Applications: strange metals and other cond-mat systems
- No Hamiltonian (bottom-up!) but spectra, conductivities...



Spectrum of heavy fermion metal



AC conductivity, 1605.07849

Fermions on global AdS

M. Milovanović

M. Dimitrijević-Ćirić

- Work in progress with M. Milovanović and M. Dimitrijević-Ćirić
- Poincare patch → global AdS: from planar QFT to a QFT on the sphere
- Sphere – compact manifold:
  - Gapped system: gap ~ $1/L$ (radius of the sphere = AdS radius) $\Longrightarrow$ regularity ensured
  - Extra parameter $\Longrightarrow$ nontrivial planar limit
  - Compactness $\Longrightarrow$ nontrivial topology

Fermions on global AdS

- Dirac spinor in the bulk:

$$S=\int d^4x\sqrt{g}\,[R-\Lambda-\bar{\Psi}(D_a\Gamma^a-m)\Psi]+S_{bnd}$$

- Strongly coupled QFT on the sphere: $\partial(AdS_4)=S^3$
- Angular momentum quantum numbers κ, m_κ
- Solution to Dirac equation: $f_\pm(z)$

$$\Psi(z)=\left(f_+(z)+\frac{\Gamma^i r_i}{r}f_-(z)\right)\chi_+\,,\qquad \chi_+ \text{ - spherical spinor}$$

- Two linearly independent branches: $f_\pm(z),\ g_\pm(z)$
- Boundary conditions:

  - IR: require finite branch only for stability

  - UV: leading term equals the source in QFT

Fermions in pure AdS

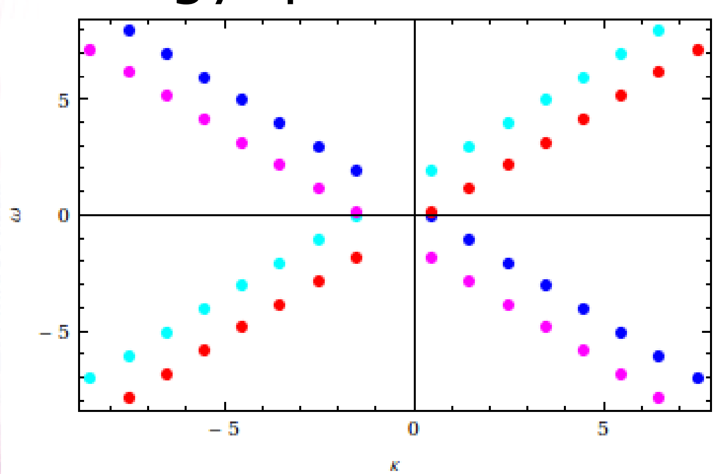
- Exact solution in pure AdS:

$$f^1_{\pm}(\theta)=(\sin\theta)^{\frac{D-1}{2}+\alpha}(\cos\theta)^{\frac{D}{2}+\beta}\;{}_2F_1\left(\alpha+\beta-\frac{\tilde{\omega}}{2},\alpha+\beta+\frac{\tilde{\omega}}{2},2\alpha+\frac{1}{2},\sin^2\theta\right)$$

$$f^2_{\pm}(\theta)=(\sin\theta)^{\frac{D+1}{2}-\alpha}(\cos\theta)^{\frac{D}{2}+\beta}\;{}_2F_1\left(\beta-\alpha-\frac{\tilde{\omega}}{2},\beta-\alpha+\frac{\tilde{\omega}}{2},\frac{3}{2}-2\alpha,\sin^2\theta\right)$$

$$\alpha_{\pm}=\frac{\kappa}{2},\frac{1+\kappa}{2},\quad \beta_{\pm}=\frac{1-m}{2},-\frac{m}{2},\quad \tilde{\omega}=\omega-\frac{1}{2}$$

- "Supersymmetric" Hamiltonian: $H^2 f=\tilde{\omega}^2 f$ (Gibbons 1990s)
- Energy quantization: finiteness of wavefunction everywhere



$D=3$

- Fermion doubling (sanity check: expected in field theory)

Finite chemical potential

- Reissner-Nordstrom metric $ds^2 = -\frac{dt^2 f(\theta)}{\cos^2\theta} + \frac{d\theta^2/f(\theta)}{\cos^2\theta} + \tan^2\theta\, d\Omega^2$
- Still supersymmetric Hamiltonian:

$$H = \begin{pmatrix} \nu & -\partial_\theta + W \\ \partial_\theta + W & -\nu \end{pmatrix} \to \begin{pmatrix} \tilde{\nu} & -\partial_\theta + W_- \\ \partial_\theta + W_+ & -\tilde{\nu} \end{pmatrix}, \quad \nu, \tilde{\nu} = const.$$

- Amazingly – analytically solvable through Whittaker functions:

$$f_\pm^1(\theta) = (\sin\theta)^{\frac{D-1}{2}+\tilde{\gamma}} (\cos\theta)^{\frac{D}{2}+\tilde{\beta}} M_{\gamma,\mu q}\left(a,b,c,\sin^2(\theta-\theta_h)^2\right)$$

$$f_\pm^2(\theta) = (\sin\theta)^{\frac{D-1}{2}-\tilde{\gamma}} (\cos\theta)^{\frac{D}{2}+\tilde{\beta}} W_{\gamma,\mu q}\left(a,b,c,\sin^2(\theta-\theta_h)^2\right)$$

$$a(\kappa,m,\mu), b(\kappa,m,\mu), c(\kappa,m,\mu), \quad \tilde{\gamma}_\pm = \sqrt{(\mu+\kappa/2)^2 \pm \kappa^2/4}, \quad \tilde{\beta}_\pm = \beta_\pm \mp M_{\%ell}$$

- Propagator (analytical without backreaction):

$$\langle \psi \bar{\psi} \rangle = \lim_{\theta \to \pi/2} \frac{\left(f_+^1(\theta)\right)^2 - \frac{(\kappa - m - \omega + 1)^2}{(2\kappa - 1)^2}\left(f_-^1(\theta)\right)^2}{(b_-)^2}$$

AdS action for a fermion

- Dirac spinor in the bulk:

$$S = \int d^4x \sqrt{g} [R - \Lambda - \bar{\Psi}(D_a \Gamma^a - m)\Psi] + S_{bnd}$$

- Radial projections (single spinors) $\Psi_\pm , \bar{\Psi}_\pm : \; \Gamma^z \Psi_\pm = \pm \Psi_\pm$
- Boundary action for on-shell stationarity (Henneaux 1997, Sfetsos 1997):

$$\delta S = \int d^4x \sqrt{g} [\delta\bar{\Psi}(D_a \Gamma^a - m)\Psi] - \frac{1}{2} \oint d^3x \sqrt{h}\, \delta\bar{\Psi}\Psi + h.c. + \delta S_{bnd}$$

$\downarrow$ $\downarrow$

$$S_{bnd} = \frac{1}{2} \oint d^{3x} \sqrt{h} \bar{\Psi}\Psi$$

- Dirac equation leaves only two independent components:

$$\Psi_+ \propto \Psi_- , \quad \bar{\Psi}_- \propto \bar{\Psi}_+$$

- Hamiltonian formalism: $\Psi_+ , \bar{\Psi}_-$ canonical momenta for $\Psi_- , \bar{\Psi}_+$

Action ambiguity

- On-shell variation: $\delta\Psi_-, \delta\bar{\Psi}_+$ arbitrary and independent, $\delta\Psi_+ = \delta\bar{\Psi}_- = 0$ as $\Psi_-, \bar{\Psi}_+$ fix $\Psi_+, \bar{\Psi}_-$
- Off-shell variation: all arbitrary, no Dirac equation $\Longrightarrow S_{bnd}$ is non-unique: $S_{bnd} \rightarrow S_{bnd} + \oint d^3x\sqrt{h}\,F(\Psi_+, \bar{\Psi}_-)$
- Bulk equation of motion unchanged by definition. What is different?
- Action changes even on-shell (solution $\psi(z_0)$):

$$S(\psi(z), \bar{\psi}(z)) = \underbrace{S_{bulk}(\psi(z), \bar{\psi}(z))}_{=0} + z_0^3\left(\frac{1}{2}\bar{\psi}(z_0)\psi(z_0) + F(\psi(z_0), \bar{\psi}(z_0))\right)$$

- Surface interaction term in the equation of motion:

$$D_a\Gamma^a\Psi_- - m\Psi_- = \sqrt{h}\frac{\delta F}{\delta\bar{\psi}_-}\delta(z)$$

F depends on $\Psi \Longrightarrow$ not a source

Adiabatic perturbation

- Idea: non-canonical term in the action only affects the subleading term in the bulk solution
- Standard AdS/CFT lore (Balasubramanian & Kraus 1999, Freedmann et al 1999):

  non-normalizable mode in AdS – source in CFT

  normalizable mode in AdS – state in CFT
- Additional term in $S \Longrightarrow$ modify field theory Lagrangian
- Dirac equation determines $\Psi_+ , \bar{\Psi}_-$ as a function of $\Psi_- , \bar{\Psi}_+$:

$$\Psi_+ = \left(e^\theta \Gamma^\theta \partial_\theta + e^i \Gamma_i \partial_i \right) \Psi_-$$

- Modify only at a single radial slice $\theta = \pi/2 - \epsilon$: consistency of the bulk Dirac equation?

Adiabatic perturbation

- Source at radial slice $\theta_0 = \pi/2 - \epsilon$ in EOM:

$$D_a \Gamma^a \Psi_- - m \Psi_- = \sqrt{h} \frac{\delta F}{\delta \bar{\Psi}_-} \delta(\theta - \theta_0)$$

- Regularity condition on F: $\frac{\delta F}{\delta \bar{\Psi}_-} \leq (\cos\theta_0)^m$
- For regular perturbations F the bulk Dirac equation conserved up to terms of order $(\cos\theta_0)^{2m}$:

$$\Psi_+ - \left(e^\theta \Gamma^\theta \partial_\theta + e^i \Gamma_i \partial_i\right) \Psi_- = \cos^{2m}\theta_0 \left(1 + c_1 \cos\theta_0 + c_2 \cos^2\theta_0 + \ldots\right)$$

Can be absorbed in regularization

- Sources in field theory: normed by $b_- (\cos\theta_0)^{-m}$ – don't change
- States in field theory – quantized energies, behave as $(\cos\theta_0)^{2m}$:

$$|n\kappa> = \frac{1}{b_- (\cos\theta_0)^{-m}} \left(f_-(\theta; \kappa, n) + \frac{\gamma^i k_i}{k} f_+(\theta; \kappa, n) \right)$$

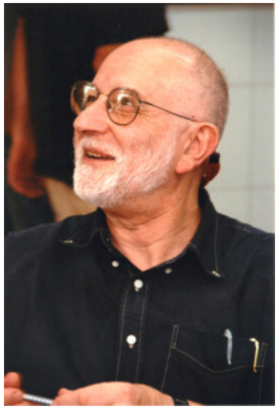
"Spin" Berry phase

- Remember the definition:

  Change the system adiabatically, stays in the same state $|n\kappa>$ but the state changes

  Along a closed path:

  $$\gamma = i \oint \langle n\kappa | \frac{\partial}{\partial\varphi} | n\kappa \rangle d\varphi$$



M. Berry

- Ansatz for F: $F(\Psi_+, \bar{\Psi}_-) = \bar{\Psi}_- e^{\frac{i\varphi\Gamma^\varphi}{2}} \Psi_+ - \bar{\Psi}_- \Psi_+$
- Rotated by Γ^z, comes back to itself after a rotation by 4π
- On the sphere, the states $|n\kappa>$ transform simply under ∂_φ, no regularization needed
- On Poincare patch we would need to introduce a regulator

"Spin" Berry phase

- Surface interaction term: $\oint d^D x \sqrt{h} \left(\bar{\Psi}_- e^{\frac{i\varphi\Gamma^\varphi}{2}} \Psi_+ - \bar{\Psi}_- \Psi_+ \right)$
- Extra surface interaction term in the Hamiltonian:

$$H = \begin{pmatrix} \nu & -\partial_\theta + W \\ \partial_\theta + W & -\nu \end{pmatrix} \qquad H_\varphi = H + \hat{s} \times \delta(\theta - \theta_0)$$

$$\hat{s} = \sin\frac{\varphi}{4} \left(\cos\frac{\varphi}{4} + \sin\frac{\varphi}{4} \begin{pmatrix} 0 & -\gamma^\varphi \\ \gamma^\varphi & 0 \end{pmatrix} \right), \quad \gamma^\varphi \chi_\pm = \pm i\, m_\kappa \chi_\pm$$

- Calculate the normalizable solution using the bulk-to-boundary propagator $G_0(\theta; \vec{x}, \vec{x}')$:

$$|n\kappa> \rightarrow |n\kappa>^\varphi \equiv |n\kappa> + \oint d^D x' \sqrt{h}\, \hat{s}(\vec{x}') G_0(\theta_0; \vec{x}, \vec{x}')$$

- Compute the Berry phase by definition
- If $m > m_c$ the terms with $\sin\frac{\varphi}{4}\cos\frac{\varphi}{4}$ give $\gamma = 0$
- If $m < m_c$ the terms with $\sin^2\frac{\varphi}{4}$ survive giving $\gamma = \pi$

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Planar limit

- From angular momentum to momentum basis
- Non-commutativity of limits:

$\lim_{L\to\infty}\gamma=\pi$ from global AdS to Poincare patch

- Ambiguity for normalizable modes on Poincare patch:

$f^1_{\pm}(z)=z^{\frac{D}{2}}J_{\nu_{\pm}}(kz),\quad f^2_{\pm}(z)=z^{\frac{D}{2}}J_{-\nu_{\pm}}(kz),\quad J_{\nu_{\pm}}$ - Bessel 1st kind

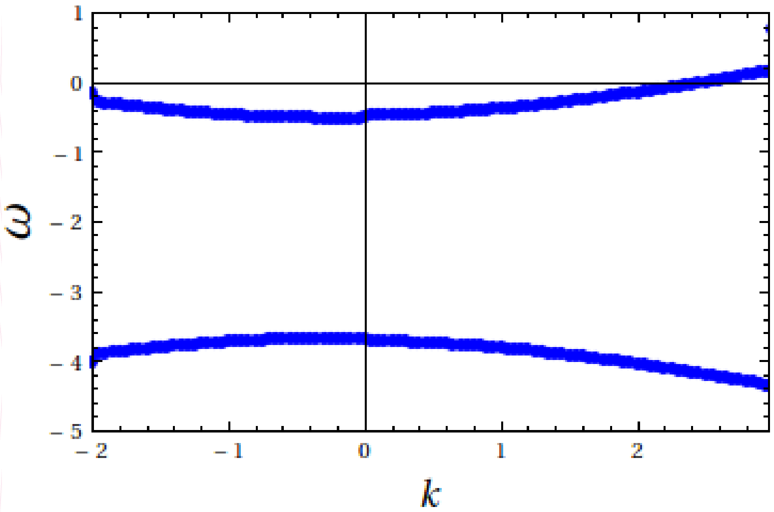
- Non-normalizable modes on Poincare patch:

$g_{\pm}(z)=z^{\frac{D}{2}}K_{\nu_{\pm}}(kz),\quad K_{\nu_{\pm}}$ - modified Bessel 1st kind

- On Poincare patch γ is regularization dependent
- Similar situation in field-theoretical derivation: at least 7 different results for the coefficient of the Chern-Simons term in literature (K. Landsteiner)

Berry phase quantization

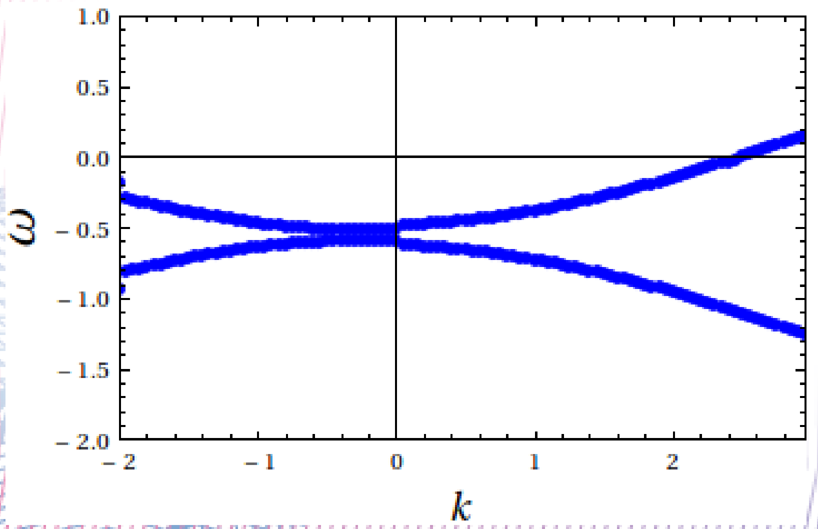
- ***Berry phase*** $\gamma=\pi$ ***for*** $\Delta<\Delta_c,\ \ \mu<1/L$



Gap survives

- ***Berry phase*** $\gamma=\pi$ ***for*** $\Delta<\Delta_c,\ \ \mu>1/L$

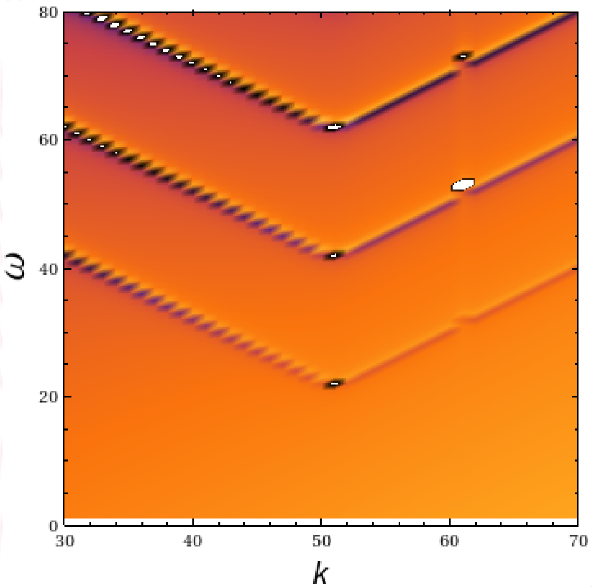
No conical singularity!



Gap closes but no singularity

Berry phase quantization

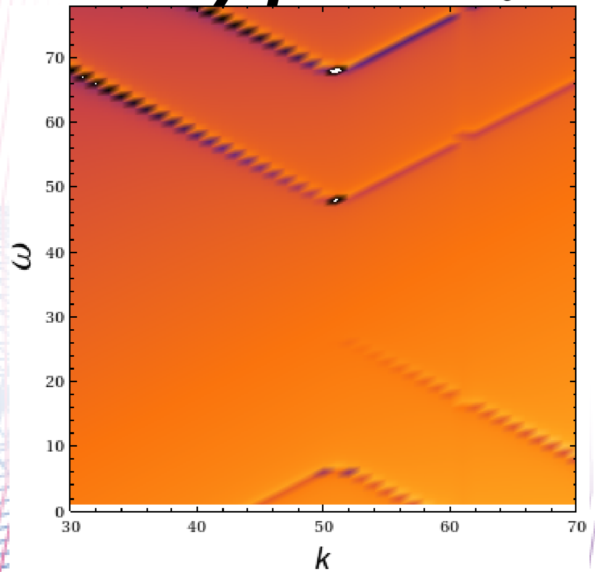
- ***Berry phase*** $\gamma=0$ ***for*** $\Delta>\Delta_c, \ \mu<1/L$



"conformal" phase:
quantum critical cones

No exact conformal invariance at finite μ

- ***Berry phase*** $\gamma=0$ ***for*** $\Delta>\Delta_c, \ \mu>1/L$



Strongly coupled Dirac liquid

- Universal Dirac cones at intermediate energies but non-universal IR physics
- Even if gapless the singularity in dispersion generically vanishes:

$$G(\omega \rightarrow 0) = \frac{1}{\omega^{\gamma} - v|k| + \ldots}$$

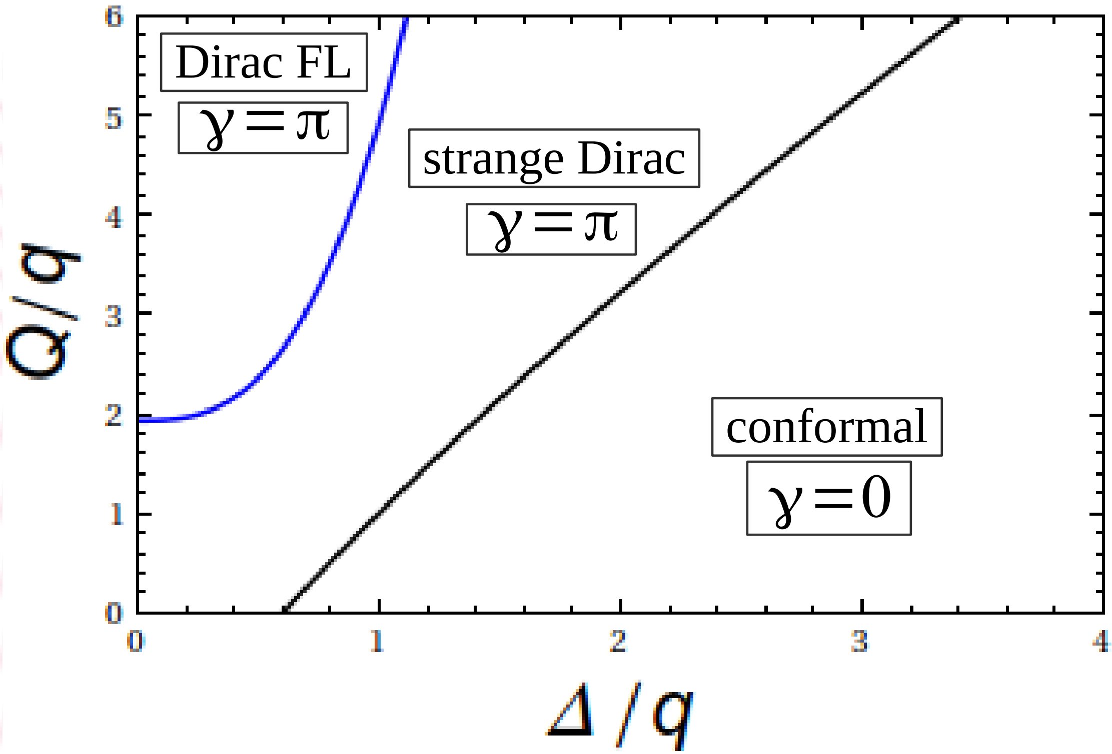
- Novel phase on the sphere: instability in IR transforms AdS into a horizon-less geometry
- Analogous to boson star solution (Gentle, Rangamani & Withers 2012)

Lesson:

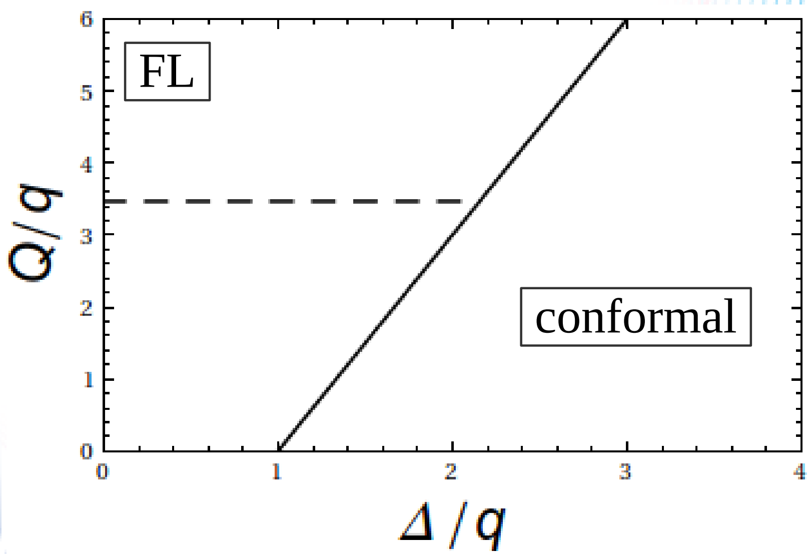
Nontrivial Berry phase does not depend on Dirac singularity

Phase diagram – planar limit

- The strange Dirac phase survives the limit!



- Compare to the Poincare patch:



Trivial Berry phase in all phases: depends on regularization

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Non-canonical Hamiltonians

- Remember how Berry phase modifies the Poisson structure:

$$\begin{pmatrix} [q_a, q_b] & [q_a, p_b] \\ [p_a, q_b] & [p_a, p_b] \end{pmatrix} = i \begin{pmatrix} i f_{ab} & \delta_{ab} \\ -\delta_{ab} & iq F_{ab} \end{pmatrix}$$

f_{ab} − Berry connection

F_{ab} − Berry connection

- Ask a more general question: is there a relation between the action ambiguity and non-canonical Poisson structures?
- In general: $(q_{1,} \dots q_n, p_{1,} \dots p_n) \rightarrow (\xi_{1,} \dots \xi_{2n})$
- Poisson structure: $\dot{\xi}_a = \omega_{ab} \frac{\partial H}{\partial \xi_b} \equiv \{\xi_a, H\}, \qquad \omega^{ab} = \{\xi^a, \xi^b\}$
- Canonical momenta not related to spatial translations (don't even exist) but H is still the generator of time translations
- Idea: modify ω^{ab} by modifying the commutator with the Hamiltonian

Operators from bulk to boundary

- Don't know the Hamiltonian explicitly! What to do?
- Bulk $so(d,2)$ symmetry algebra: D, P_a, K_a, M_{ab}
- On AdS$_3$ (2D QFTs) decouple left- and right-movers:

$$K^{\pm} = \frac{1}{2}(K_1 \pm i K_2), \quad P^{\pm} = \frac{1}{2}(P_1 \mp i P_2), \quad L^{\pm} = D \mp M_{12}$$

- QFT Hamiltonian from L and $D = -i\partial_t$
- Modify the generators but keep the conformal algebra at leading order:

$$(L, P, K) \rightarrow (L + L_1, P + P_1, K + K_1)$$

$$[K, L_1] + [K_1, L] = K_1$$

$$[K, P_1] + [K_1, P] = 2L_1$$

$$[L_1, P] + [L, P_1] = P_1$$

Operators from bulk to boundary

- Restrict the action of $K^{\pm}$, $P^{\pm}$, $L^{\pm}$ on the AdS boundary:

  $k^{\pm}=\lim_{\theta\to\pi/2}K^{\pm}$, etc.

- Compansate for the action of L_1 by S_1:

  $$\frac{\delta S}{\delta\Phi}=[S,L]=[S,L]+[S_{1,}L]+[S,L_1]\Leftrightarrow[S_{1,}L]+[S,L_1]=0$$

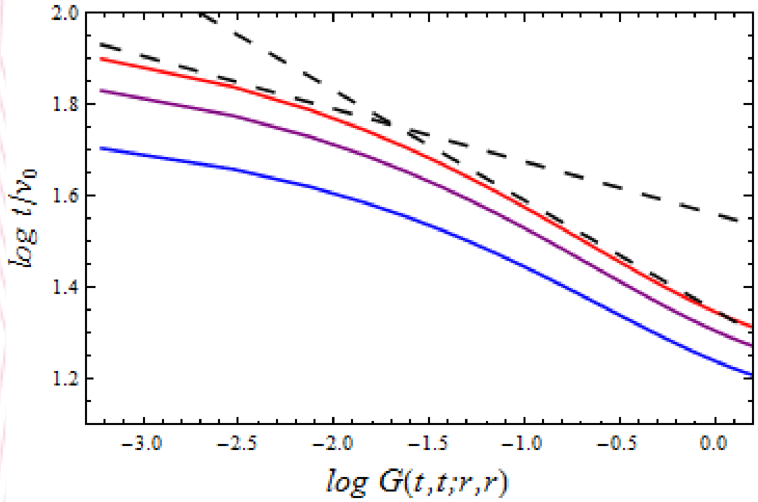
- Drawback: need guesswork for S_1
- For the Berry phase: $S_1=\oint d^D x\sqrt{h}\,\Pi_1 e^{i\hat{f}/2}\Pi_2$
- Canonical coordinates $(q_{1,}q_2)=\Psi_-,\bar{\Psi}_+$, canonical momenta $(\Pi_{1,}\Pi_2)=\Psi_+,\bar{\Psi}_-$
- Still ambigous what to pick for $\hat{f}$ - need physical input?

Fluid advection

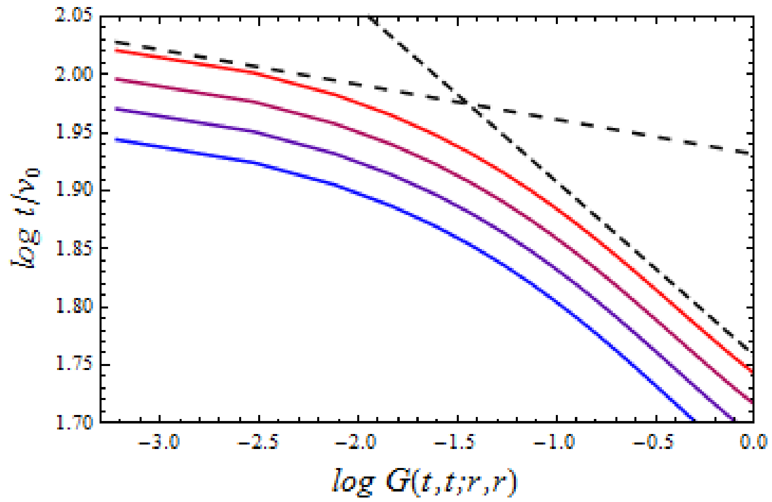
- Fluid dynamics – Eulerian vs. Lagrangian
- Lagrangian dynamics = passive scalar (tracer) advection
- Hamiltonian: $H=\frac{1}{2}\rho u^2+U\rho$
- More convenient than Eulerian approach for correlation decay etc
- Poisson bracket – add extra term F in the boundary action
- Now F is a complicated series in derivatives of the metric

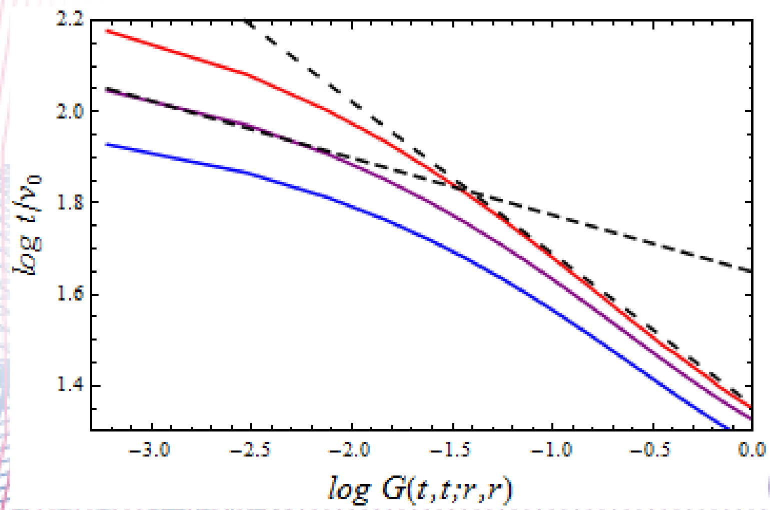
Correlation decay

- Inhomogenuous distributions, dynamics $\Longrightarrow$ numerics



- Initial local decay
- Long-time faster hydrodynamic tail





Slow non-universal decay:

$G \sim 1/t^{\nu}, \quad \nu \approx 0.2$

Conclusions

Physical lesson

- Divorce the Berry phase from Dirac fermion physics
- Dirac singularity generically absent in the presence of interaction but nontrivial Berry phase is there

Formal lesson

- "Pen&paper" work still remains in phenomenological AdS/CFT
- Understand better the meaning of action ambiguity from top-down setups