Birefringence property of the Moyal-Weyl noncommutative spacetime in $SO(2,3)_\star$ model

Dragoljub Gočanin

Faculty of Physics, University of Belgrade

2018 Workshop on Gravity, Holography, Strings and Noncommutative Geometry

Dragoljub Gočanin, Voja Radovanović, Dirac field and gravity in NC $SO(2,3)_\star$ model,
*arXiv:*1708.07437

February 1, 2018

Introduction

In the last twenty years Noncommutative (NC) Field Theory has become a significant direction of investigation in gravitational and theoretical high energy physics.

Basic insight: *complementarity* of different spacetime dimensions $\rightarrow$ *quantization by deformation*

Canonical noncommutativity: $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$ where $\theta^{\mu\nu}$ are components of a *constant* antisymmetric matrix.

Alternative way: deform the *algebra of functions* (fields) on spacetime by introducing NC Moyal-Weyl $\star$-product,

$$(\hat{f} \star \hat{g})(x) = e^{\frac{i}{2}\theta^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial y^\beta}} f(x)g(y)|_{y\rightarrow x} \qquad (1)$$

and use ordinary commutative co-ordinates.

Motivation

- What type of couplings between matter and gravity arise in NC theory?
- How noncommutativity "deforms" the potentially observable physics, e.g. electron's dispersion relation?
- Answering these questions within a unifying framework for gauge field theories and gravity with an enlarged gauge group; $SO(2,3)$ model.

Previous studies of the subject have resulted in:

1. Construction of the consistent model for deformed Einstein gravity including the analysis of the NC equations of motion.
2. Revealing the physical meaning for constant noncommutativity - Fermi normal coordinates.

Next step: incorporating matter fields, specifically, the Dirac spinor field, in $SO(2,3)_\star$ model.

AdS algebra

Generators of $SO(2,3)$ gauge group satisfy

$$[M_{AB}, M_{CD}] = i(\eta_{AD}M_{BC} + \eta_{BC}M_{AD} - \eta_{AC}M_{BD} - \eta_{BD}M_{AC}) \quad (2)$$

where $\eta_{AB} = \mathrm{diag}(+,-,-,-,+)$ is the 5D internal space metric and group indices A, B, ... take values $0,1,2,3,5$. In terms of 5D gamma matrices that satisfy the usual anticommutation relations

$$\{\Gamma^A, \Gamma^B\} = 2\eta^{AB} \quad (3)$$

they are given by

$$M_{AB} = \frac{i}{4}[\Gamma_A, \Gamma_B] \quad (4)$$

One choice is $\Gamma_A = (i\gamma_a\gamma_5, \gamma_5)$ where $a = 0,1,2,3$. In this particular representation, $SO(2,3)$ generators are

$$M_{ab} = \frac{1}{2}\sigma_{ab} \qquad M_{5a} = \frac{1}{2}\gamma_a \quad (5)$$

Gauge fields

The total $SO(2,3)$ gauge potential can be decomposed as

$$\omega_\mu = \frac{1}{2}\omega_\mu{}^{AB} M_{AB} = \frac{1}{4}\omega_\mu{}^{ab}\sigma_{ab} - \frac{1}{2l} e^a_\mu \gamma_a \tag{6}$$

where e^a_μ and $\omega_\mu{}^{ab}$ are the **vierbein** and the $SO(1,3)$ **spin-connection** and l is a constant length scale. The world indices $\mu, \nu, \ldots$ take values $0,1,2,3$.

The field strength tensor

$$F_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu - i[\omega_\mu, \omega_\nu] = \frac{1}{2} F_{\mu\nu}{}^{AB} M_{AB} \tag{7}$$

has components

$$F_{\mu\nu}{}^{ab} = R_{\mu\nu}{}^{ab} - \frac{1}{l^2}(e^a_\mu e^b_\nu - e^b_\mu e^a_\nu) \qquad F_{\mu\nu}{}^{a5} = \frac{1}{l} T_{\mu\nu}{}^{a} \tag{8}$$

where

$$R_{\mu\nu}{}^{ab} = \partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab} + \omega_\mu{}^{ac}\omega_\nu{}^{cb} - \omega_\mu{}^{bc}\omega_\nu{}^{ca} \tag{9}$$

$$T_{\mu\nu}{}^{a} = \nabla_\mu e^a_\nu - \nabla_\nu e^a_\mu \tag{10}$$

Commutative model

The kinetic term

$$S_{kin} = \frac{i}{12}\int d^4x\, \varepsilon^{\mu\nu\rho\sigma}\left[\bar{\psi} D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\psi - D_\sigma\bar{\psi} D_\mu\phi D_\nu\phi D_\rho\phi\psi\right] \quad (11)$$

Covariant derivative of a Dirac spinor is

$$D_\sigma\psi = \partial_\sigma\psi - \frac{i}{2}\omega_\sigma{}^{AB} M_{AB}\psi = \nabla_\sigma\psi + \frac{i}{2l} e^a_\sigma \gamma_a\psi \quad (12)$$

where

$$\nabla_\sigma\psi = \partial_\sigma\psi - \frac{i}{4}\omega_\sigma{}^{ab}\sigma_{ab}\psi \quad (13)$$

is the usual $SO(1,3)$ covariant derivative.

Adjoint auxiliary field $\phi = \phi^A\Gamma_A$ is constrained by $\phi^A\phi_A = l^2$.

Covariant derivative: $D_\mu\phi = \partial_\mu\phi - i[\omega_\mu, \phi]$

Symmetry breaking

In order to break the symmetry from $SO(2,3)$ to $SO(1,3)$ we fix the value of the auxiliary field: $\phi^a = 0$ and $\phi^5 = l$.

Components of the covariant derivative are $(D_\mu\phi)^a = e^a_\mu$ and $(D_\mu\phi)^5 = 0$.

This gives us

$$S_{kin} = \frac{i}{2}\int d^4x\ e\ \left[\bar{\psi}\gamma^\sigma\nabla_\sigma\psi - \nabla_\sigma\bar{\psi}\gamma^\sigma\psi\right] - \frac{2}{l}\int d^4x\ e\ \bar{\psi}\psi \qquad (14)$$

i.e. the Dirac action in curved spacetime for fermions of mass $2/l$.

We need additional mass terms:

$$\begin{aligned} S_{m,1} &= \frac{i}{2}c_1\left(\frac{m}{l} - \frac{2}{l^2}\right)\int d^4x\ \varepsilon^{\mu\nu\rho\sigma}\bar{\psi}D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\phi\phi\psi + h.c. \\ S_{m,2} &= \frac{i}{2}c_2\left(\frac{m}{l} - \frac{2}{l^2}\right)\int d^4x\ \varepsilon^{\mu\nu\rho\sigma}\bar{\psi}D_\mu\phi D_\nu\phi D_\rho\phi\phi D_\sigma\phi\psi + h.c. \\ S_{m,3} &= i\ c_3\left(\frac{m}{l} - \frac{2}{l^2}\right)\int d^4x\ \varepsilon^{\mu\nu\rho\sigma}\bar{\psi}D_\mu\phi D_\nu\phi\phi D_\rho\phi D_\sigma\phi\psi \end{aligned} \qquad (15)$$

Constraint: $c_2 - c_1 - c_3 = -\frac{1}{24}$

NC deformation

Substitute ordinary point-wise multiplication with NC Moyal-Weyl $\star$-product in "symmetric-phase" action. The obtained NC action is hermitian and invariant under local $SO(2,3)_\star$ transformations.

$$\widehat{S} = \frac{i}{12}\int d^4x\, \varepsilon^{\mu\nu\rho\sigma}\, \Big[\widehat{\bar{\psi}}\star(D_\mu\widehat{\phi})\star(D_\nu\widehat{\phi})\star(D_\rho\widehat{\phi})\star(D_\sigma\widehat{\psi}) - (D_\sigma\widehat{\bar{\psi}})\star(D_\mu\widehat{\phi})\star(D_\nu\widehat{\phi})\star(D_\rho\widehat{\phi})\star\widehat{\psi}\Big] + \widehat{S}_m \qquad (16)$$

Seiberg-Witten map - a way to represent NC fields as a perturbative series in powers of $\theta^{\alpha\beta}$ with coefficients built out of the ordinary fields; there are no new degrees of freedom (no new fields) in the NC theory, just new interactions. For example, SW for NC fields $\widehat{\phi}$ and $\widehat{\psi}$ is

$$\widehat{\phi} = \phi - \frac{1}{4}\theta^{\alpha\beta}\{\omega_\alpha, (\partial_\beta + D_\beta)\phi\} + \mathcal{O}(\theta^2) \qquad (17)$$

$$\widehat{\psi} = \psi - \frac{1}{4}\theta^{\alpha\beta}\omega_\alpha(\partial_\beta + D_\beta)\psi + \mathcal{O}(\theta^2) \qquad (18)$$

The expansion rule

$$\left(\widehat{A}\star\widehat{B}\right)^{(1)}=\widehat{A}^{(1)}B+A\widehat{B}^{(1)}+\frac{i}{2}\theta^{\alpha\beta}\partial_\alpha A\partial_\beta B \tag{19}$$

If both of these two fields transform in the *adjoint representation* we have

$$\begin{aligned}\left(\widehat{A}\star\widehat{B}\right)^{(1)}=&-\frac{1}{4}\theta^{\alpha\beta}\{\omega_\alpha,(\partial_\beta+D_\beta)AB\}+\frac{i}{2}\theta^{\alpha\beta}D_\alpha AD_\beta B\\&+cov(\widehat{A}^{(1)})B+Acov(\widehat{B}^{(1)})\end{aligned} \tag{20}$$

$$D_\mu\widehat{\phi}=D_\mu\phi-\frac{1}{4}\theta^{\alpha\beta}\{\omega_\alpha,(\partial_\beta+D_\beta)D_\mu\phi\}+\frac{1}{2}\theta^{\alpha\beta}\{F_{\alpha\mu},D_\beta\phi\}+\mathcal{O}(\theta^2) \tag{21}$$

$$\begin{aligned}\left(D_\mu\widehat{\phi}\star D_\nu\widehat{\phi}\right)^{(1)}=&-\frac{1}{4}\theta^{\alpha\beta}\{\omega_\alpha,(\partial_\beta+D_\beta)D_\mu\phi D_\nu\phi\}+\frac{i}{2}\theta^{\alpha\beta}D_\alpha D_\mu\phi D_\beta D_\nu\phi\\&+\frac{1}{2}\theta^{\alpha\beta}\{F_{\alpha\mu},D_\beta\phi\}D_\nu\phi+\frac{1}{2}\theta^{\alpha\beta}D_\mu\phi\{F_{\alpha\nu},D_\beta\phi\}\end{aligned} \tag{22}$$

NC kinetic term at the first order

$$
\begin{aligned}
\widehat{S}^{(1)}_{kin} = \frac{i}{12}\,\theta^{\alpha\beta}\int d^4x\;\varepsilon^{\mu\nu\rho\sigma}\Bigg[& -\frac{1}{4}\bar{\psi}F_{\alpha\beta}D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\psi \\
& + \frac{i}{2}\bar{\psi}D_\alpha(D_\mu\phi D_\nu\phi D_\rho\phi)(D_\beta D_\sigma\psi) \\
& + \frac{i}{2}\bar{\psi}D_\alpha(D_\mu\phi D_\nu\phi)(D_\beta D_\rho\phi)D_\sigma\psi \\
& + \frac{i}{2}\bar{\psi}(D_\alpha D_\mu\phi)(D_\beta D_\nu\phi)D_\rho\phi D_\sigma\psi \\
& + \frac{1}{2}\bar{\psi}\{F_{\alpha\mu}, D_\beta\phi\}D_\nu\phi D_\rho\phi D_\sigma\psi \\
& + \frac{1}{2}\bar{\psi}D_\mu\phi\{F_{\alpha\nu}, D_\beta\phi\}D_\rho\phi D_\sigma\psi \\
& + \frac{1}{2}\bar{\psi}D_\mu\phi D_\nu\phi\{F_{\alpha\rho}, D_\beta\phi\}D_\sigma\psi \\
& - \frac{1}{2}\bar{\psi}D_\mu\phi D_\nu\phi D_\rho\phi F_{\sigma\alpha}D_\beta\psi \Bigg] + h.c. \qquad (23)
\end{aligned}
$$

Linear NC correction to the Dirac action in curved spacetime

$$
\begin{aligned}
\widehat{S}^{(1)}_{kin} = \theta^{\alpha\beta}\Big[\quad & - \frac{1}{8}\int d^4x\ e\ R_{\alpha\mu}{}^{ab} e^{\mu}_{a}\ \bar{\psi}\gamma_b \nabla_\beta \psi + \frac{1}{16}\int d^4x\ e\ R_{\alpha\beta}{}^{ab} e^{\sigma}_{b}\ \bar{\psi}\gamma_a \nabla_\sigma \psi \\
& - \frac{i}{32}\int d^4x\ e\ R_{\alpha\beta}{}^{ab} \varepsilon_{abc}{}^{d} e^{\sigma}_{d}\ \bar{\psi}\gamma^c\gamma^5 \nabla_\sigma \psi - \frac{i}{16}\int d^4x\ e\ R_{\alpha\mu}{}^{bc} e^{\mu}_{a} \varepsilon^{a}{}_{bcm}\ \bar{\psi}\gamma^m\gamma^5 \nabla_\beta \psi \\
& - \frac{i}{24}\int d^4x\ e\ R_{\alpha\mu}{}^{ab} \varepsilon_{abc}{}^{d} e^{c}_{\beta} (e^{\mu}_{d} e^{\sigma}_{s} - e^{\mu}_{s} e^{\sigma}_{d})\ \bar{\psi}\gamma^s\gamma^5 \nabla_\sigma \psi \\
& - \frac{i}{8l}\int d^4x\ e\ T_{\alpha\beta}{}^{a} e^{\sigma}_{a}\ \bar{\psi}\nabla_\sigma \psi + \frac{i}{8l}\int d^4x\ e\ T_{\alpha\mu}{}^{a} e^{\mu}_{a}\ \bar{\psi}\nabla_\beta \psi \\
& + \frac{1}{16l}\int d^4x\ e\ T_{\alpha\beta}{}^{a} e^{\mu}_{a}\ \bar{\psi}\sigma_{\mu}{}^{\sigma} \nabla_\sigma \psi + \frac{1}{8l}\int d^4x\ e\ T_{\alpha\mu}{}^{a} e^{\mu}_{b}\ \bar{\psi}\sigma_{a}{}^{b} \nabla_\beta \psi \\
& - \frac{1}{12l}\int d^4x\ e\ T_{\alpha\mu}{}^{a} \varepsilon_{ab}{}^{cd} e^{b}_{\beta} e^{\mu}_{c} e^{\sigma}_{d}\ \bar{\psi}\gamma^5 \nabla_\sigma \psi + \frac{7i}{48l^2}\int d^4x\ e\ \varepsilon_{abc}{}^{d} e^{a}_{\alpha} e^{b}_{\beta} e^{\sigma}_{d}\ \bar{\psi}\gamma^c\gamma^5 \nabla_\sigma \psi \\
& - \frac{1}{4}\int d^4x\ e\ (\nabla_\alpha e^{a}_{\mu})(e^{\mu}_{a} e^{\sigma}_{b} - e^{\sigma}_{a} e^{\mu}_{b})\ \bar{\psi}\gamma^b \nabla_\beta \nabla_\sigma \psi - \frac{1}{4l}\int d^4x\ e\ \bar{\psi}\sigma_{\alpha}{}^{\sigma} \nabla_\beta \nabla_\sigma \psi \\
& - \frac{i}{8}\int d^4x\ e\ \eta_{ab}(\nabla_\alpha e^{a}_{\mu})(\nabla_\beta e^{b}_{\nu})\varepsilon^{cdrs} e^{\mu}_{c} e^{\nu}_{d} e^{\sigma}_{s}\ \bar{\psi}\gamma_r\gamma_5 \nabla_\sigma \psi \\
& + \frac{i}{12}\int d^4x\ e\ (\nabla_\alpha e^{a}_{\mu})(\nabla_\beta e^{b}_{\nu})\varepsilon_{b}{}^{cds} e^{\mu}_{c} e^{\nu}_{d} e^{\sigma}_{s}\ \bar{\psi}\gamma_a\gamma_5 \nabla_\sigma \psi \\
& - \frac{1}{12l}\int d^4x\ e\ e^{c}_{\alpha}(\nabla_\beta e^{b}_{\nu})\varepsilon_{bc}{}^{ds} e^{\nu}_{d} e^{\sigma}_{s}\ \bar{\psi}\gamma_5 \nabla_\sigma \psi \\
& - \frac{1}{8l}\int d^4x\ e\ (\nabla_\alpha e^{a}_{\mu})(e^{\mu}_{a} e^{\sigma}_{b} - e^{\sigma}_{a} e^{\mu}_{b}) e^{c}_{\beta}\ \bar{\psi}\sigma^{b}{}_{c} \nabla_\sigma \psi
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{1}{96l}\int d^4x\ e\ R_{\alpha\beta}{}^{ab}\ \bar\psi\sigma_{ab}\psi - \frac{1}{3l^3}\int d^4x\ e\ \bar\psi\sigma_{\alpha\beta}\psi \\
&- \quad \frac{5}{48l}\int d^4x\ e\ R_{\alpha\mu}{}^{ab}e_a^\mu e_\beta^c\ \bar\psi\sigma_{bc}\psi - \frac{1}{16l}\int d^4x\ e\ R_{\alpha\mu}{}^{ab}e_{\beta a}e_c^\mu\ \bar\psi\sigma_b{}^c\psi \\
&- \quad \frac{3}{32l^2}\int d^4x\ e\ T_{\alpha\beta}{}^a\ \bar\psi\gamma_a\psi - \frac{1}{16l^2}\int d^4x\ e\ T_{\alpha\mu}{}^a e_a^\mu\ \bar\psi\gamma_\beta\psi \\
&+ \quad \frac{1}{16l^2}\int d^4x\ e\ T_{\alpha\mu}{}^a e_{\beta a}\ \bar\psi\gamma^\mu\psi + \frac{1}{12l}\int d^4x\ e\ \eta_{ab}(\nabla_\alpha e_\mu^a)(\nabla_\beta e_\nu^b)\ \bar\psi\sigma^{\mu\nu}\psi \\
&- \quad \frac{1}{6l}\int d^4x\ e\ (\nabla_\alpha e_\mu^a)(\nabla_\beta e_\nu^b)(e_a^\mu e_c^\nu - e_c^\mu e_a^\nu)\ \bar\psi\sigma^c{}_b\psi \\
&- \quad \frac{3}{16l^2}\int d^4x\ e\ (\nabla_\alpha e_\mu^a)e_a^\mu\ \bar\psi\gamma_\beta\psi + \frac{1}{16l^2}\int d^4x\ e\ (\nabla_\alpha e_\mu^a)e_{\beta a}\ \bar\psi\gamma^\mu\psi \quad \Bigg] + h.c.
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
\widehat{S}_m^{(1)} = \theta^{\alpha\beta}\Bigg[&- \quad \frac{i}{4}\left(m-\frac{2}{l}\right)\int d^4x\ e\ (\nabla_\alpha e_\mu^a)e_a^\mu\ \bar\psi\nabla_\beta\psi \\
&+ \quad \frac{1}{24}\left(m-\frac{2}{l}\right)\int d^4x\ e\ \eta_{ab}(\nabla_\alpha e_\mu^a)(\nabla_\beta e_\nu^b)\ \bar\psi\sigma^{\mu\nu}\psi \\
&- \quad \frac{1}{12}\left(m-\frac{2}{l}\right)\int d^4x\ e\ (\nabla_\alpha e_\mu^a)(\nabla_\beta e_\nu^b)(e_a^\mu e_c^\nu - e_c^\mu e_a^\nu)\ \bar\psi\sigma^c{}_b\psi \\
&- \quad \frac{1}{36}\left(\frac{m}{l}-\frac{2}{l^2}\right)\int d^4x\ e\ (\nabla_\alpha e_\mu^a)e_a^\mu\ \bar\psi\gamma_\beta\psi \\
&- \quad \frac{1}{96}\left(m-\frac{2}{l}\right)\int d^4x\ e\ R_{\alpha\beta}{}^{ab}\ \bar\psi\sigma_{ab}\psi - \frac{1}{12}\left(m-\frac{2}{l}\right)\int d^4x\ e\ R_{\alpha\mu}{}^{ab}e_a^\mu e_\beta^c\ \bar\psi\sigma_{bc}\psi \\
&- \quad \frac{1}{72}\left(\frac{m}{l}-\frac{2}{l^2}\right)\int d^4x\ e\ T_{\alpha\beta}{}^a\ \bar\psi\gamma_a\psi - \frac{7}{72}\left(\frac{m}{l}-\frac{2}{l^2}\right)\int d^4x\ e\ T_{\alpha\mu}{}^a e_a^\mu\ \bar\psi\gamma_\beta\psi \\
&- \quad \frac{1}{8}\left(\frac{m}{l^2}-\frac{2}{l^3}\right)\int d^4x\ e\ \bar\psi\sigma_{\alpha\beta}\psi \quad \Bigg] + h.c.
\end{aligned}
\tag{25}
$$

Flat spacetime

Deformed Dirac action

$$\widehat{S} = \int d^4x\, \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$
$$+\theta^{\alpha\beta}\int d^4x \left[-\frac{1}{2l}\bar{\psi}\sigma_\alpha{}^\sigma\partial_\beta\partial_\sigma\psi + \frac{7i}{24l^2}\varepsilon_{\alpha\beta}{}^{\rho\sigma}\bar{\psi}\gamma_\rho\gamma_5\partial_\sigma\psi - M\bar{\psi}\sigma_{\alpha\beta}\psi\right] + \mathcal{O}(\theta^2) \tag{26}$$

with $M := \frac{m}{4l^2} + \frac{1}{6l^3}$.

Deformed Dirac equation

$$\left[i\partial\!\!\!/ - m - \frac{1}{2l}\theta^{\alpha\beta}\sigma_\alpha{}^\sigma\partial_\beta\partial_\sigma + \frac{7i}{24l^2}\theta^{\alpha\beta}\varepsilon_{\alpha\beta}{}^{\rho\sigma}\gamma_\rho\gamma_5\partial_\sigma - \theta^{\alpha\beta}M\sigma_{\alpha\beta}\right]\psi = 0 \tag{27}$$

Assume just $[\hat{x}^1, \hat{x}^2] = i\theta^{12}$, i.e. $\theta^{12} = -\theta^{21} \equiv \theta$, and all the other equal to zero.

$$\left[i\partial\!\!\!/ - m - \frac{\theta}{2l}(\sigma_1{}^\sigma\partial_2\partial_\sigma - \sigma_2{}^\sigma\partial_1\partial_\sigma) + \frac{7i\theta}{12l^2}(\gamma_0\gamma_5\partial_3 - \gamma_3\gamma_5\partial_0) - 2\theta M\sigma_{12}\right]\psi = 0 \tag{28}$$

Dispersion relation

Since hamiltonian commutes with the whole momentum operator, we can assume the *plane wave ansatz* $\psi(x) = u(\mathbf{p})e^{-ip\cdot x}$. For simplicity we assume $\mathbf{p} = (0, 0, p_z)$ and obtain

$$\begin{pmatrix} E - m + \theta A & 0 & -p_z - \frac{7\theta}{12l^2}p_z & 0 \\ 0 & E - m - \theta A & 0 & p_z - \frac{7\theta}{12l^2}p_z \\ p_z + \frac{7\theta}{12l^2}p_z & 0 & -E - m + \theta B & 0 \\ 0 & -p_z + \frac{7\theta}{12l^2}p_z & 0 & -E - m - \theta B \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 0 \qquad (29)$$

with

$$A = \frac{7E}{12l^2} - 2M \qquad B = -\frac{7E}{12l^2} - 2M \qquad (30)$$

Energy should also be represented as a power series in θ, i.e.

$$E = \sum_{n=0}^{+\infty} E^{(n)} \ , \quad \text{where} \quad E^{(n)} \sim \frac{\theta^n}{(length)^{2n+1}} \qquad (31)$$

Dispersion relation

From the condition $det = 0$ we get four different solutions for the energy

$$\begin{aligned} E_{1,2} &= E_{\mathbf{p}} \mp \left[\frac{m^2}{12l^2} - \frac{m}{3l^3}\right]\frac{\theta}{E_{\mathbf{p}}} + \mathcal{O}(\theta^2) \\ E_{3,4} &= -E_{\mathbf{p}} \pm \left[\frac{m^2}{12l^2} - \frac{m}{3l^3}\right]\frac{\theta}{E_{\mathbf{p}}} + \mathcal{O}(\theta^2) \end{aligned} \qquad (32)$$

with $E_{\mathbf{p}} = \sqrt{m^2 + p_z^2}$. This is reminiscent of the well known **Zeeman effect** with θ playing the role of a constant background magnetic field that causes the splitting of atomic energy levels.

The noncommutativity of space, here taken to be confined in x, y-plane, causes the undeformed energy levels $\pm E_{\mathbf{p}}$ to split. The energy gap between the new levels is the same for $\pm E_{\mathbf{p}}$ and it equals

$$2\left[\frac{m^2}{12l^2} - \frac{m}{3l^3}\right]\frac{\theta}{E_{\mathbf{p}}} \qquad (33)$$

Solutions

As oppose to the commutative case, in NC theory *states with opposite helicity have different energies*!

$$\psi_1 \sim \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E_{\mathbf{p}}+m}\left[1+\left(\frac{m}{12 l^2}-\frac{1}{3 l^3}\right)\frac{\theta}{E_{\mathbf{p}}}\right] \\ 0 \end{pmatrix} e^{-iE_1 t + ip_z z} \qquad \psi_2 \sim \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{p_z}{E_{\mathbf{p}}+m}\left[1-\left(\frac{m}{12 l^2}-\frac{1}{3 l^3}\right)\frac{\theta}{E_{\mathbf{p}}}\right] \end{pmatrix} e^{-iE_2 t - ip_z z}$$

$$\psi_3 \sim \begin{pmatrix} \frac{p_z}{E_{\mathbf{p}}+m}\left[1+\left(\frac{m}{12 l^2}-\frac{1}{3 l^3}\right)\frac{\theta}{E_{\mathbf{p}}}\right] \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{-iE_3 t - ip_z z} \qquad \psi_4 \sim \begin{pmatrix} 0 \\ \frac{p_z}{E_{\mathbf{p}}+m}\left[1-\left(\frac{m}{12 l^2}-\frac{1}{3 l^3}\right)\frac{\theta}{E_{\mathbf{p}}}\right] \\ 0 \\ 1 \end{pmatrix} e^{-iE_4 t + ip_z z}$$

Birefringence

From the dispersion relations we find the (group) velocity of an electron

$$\mathbf{v} \equiv \frac{\partial E}{\partial \mathbf{p}} \tag{34}$$

For positive (negative) helicity solution ψ_1 (ψ_2) we get

$$\mathbf{v}_{1,2} = \frac{\mathbf{p}}{E_{\mathbf{p}}} \left[1 \pm \left(\frac{m^2}{12 l^2} - \frac{m}{3 l^3} \right) \frac{\theta}{E_{\mathbf{p}}^2} + \mathcal{O}(\theta^2) \right] \tag{35}$$

Thus, we conclude that velocity of an electron moving in z-direction *depends on its helicity*. This is analogues to the **birefringence effect**, i.e. the optical property of a material having a refractive index that depends on the polarization of light. NC background acts as a *birefringent medium* for electrons propagating in it.

Thank you for your attention!