Thermodynamic instabilities of extremal black holes

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Outline

- Motivation - phase transitions of the extremal black hole
- Method - entropy function formalism
- System - charged black hole coupled to the scalar field in 4D
- Results - critical behavior of the extremal black hole
- Conclusions

Motivation

- **Free energy** of the black hole carries an important information about its thermodynamic stability.
- **Thermal phase transitions** arise due to fluctuations of the temperature of black hole, when a new phase has smaller free energy. For example, BH can develop hair below some T_c.
- Using the **AdS/CFT correspondence**, a dual theory can describe a phase transition in condensed matter physics, such as holographic superconductors [*e.g.*, Hartnoll, Herzog Horowitz 2008]
- An **equilibrium state** of a thermodynamic system corresponds to a minimum of the internal energy in the energy representation of states, or a maximum of the entropy in the entropy representation of states.

Motivation

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- **Entropy** of extremal black hole arises due to a degenerate quantum ground state; it is a suitable quantity for studying its equilibria.
- **Quantum phase transition** arises due to quantum fluctuations, which produce instabilities of the system around a critical point. This phenomenon is known in the physics of condensed matter (*spin glasses*).
- Magnetized black hole close to extremality can exhibit the Meissner effect typical for phase transitions [Astorino 2015].
- Previous examples show that extremal black holes can have phase transitions, too.
- **Macroscopic entropy** of the extremal black hole can be calculated using the entropy function formalism [Sen 2005].

Motivation

- **Isometries of a near-horizon geometry of an extremal BH in 4D**:

- spherically symmetric ($AdS_2 \otimes \mathbb{S}^2$ geometry)$\rightarrow$ $SO(2,1) \otimes SO(3)$;
- rotating ($AdS_2 \otimes \mathbb{S}^1$ geometry)$\rightarrow$ $SO(2,1) \otimes U(1)$;
- topological ($AdS_2 \otimes \mathbb{S}^2$, $AdS_2 \otimes \mathbb{H}^2$, $AdS_2 \otimes \mathbb{R}^2$ geometries)
  $\rightarrow$ $SO(2,1) \otimes SO(3)$, $SO(2,1) \otimes SO(2,1)$, $SO(2,1) \otimes \mathbb{R}^2$;

- Can be generalized to other extremal geometries, such as warped ones [Astefanesei, Miskovic, Olea 2012].

- The accelerating Kerr-Newman geometry can be included, too, by allowing warping and twisting of the above geometies [Astorino 2015, 2016].

- The black hole horizon always has AdS_2 geometry.

Motivation

- **Our interest – gravity with the cosmological constant**

- Horizon geometry of an extremal BH is $\mathrm{AdS}_2 \otimes \Sigma_k$

- 2D transversal section Σ_k can be a $\begin{cases} \text{2-sphere } (\mathbb{S}^2) & k = 1 \\ \text{2D plane } (\mathbb{R}^2) & k = 0 \\ \text{2-hyperboloid } (\mathbb{H}^2) & k = -1 \end{cases}$

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- **Entropy function formalism** deals with near-horizon parameters only, seeing the action as a function of the charges, scalar fields and the parameters of the near-horizon geometry.
- It is based on a **variational principle** applied to a generic class of actions.
- Extremization of the entropy function determines **all the near-horizon parameters** without knowledge of a particular solution.

Motivation

- **Some results about the horizon instabilities**:

- A massless scalar field produces an instability at the horizon of an extreme RN BH (because the energy density degenerates on the horizon when studying the dispersion of the wave equation on the BH spacetime) [Aretakis 2013]

- The axisymmetric extremal horizons are unstable under linear scalar perturbations [Aretakis 2015], [Lucietti, Murata, Reall 2013]

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- **Non-extremal BH**: Stückelberg scalar has been known to describe both first and second order thermal phase transitions. A question is whether a similar change would also occur at $T = 0$. [Hartnoll, Herzog Horowitz 2008], [Franco, García-García, Rodríguez-Gómez 2009

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- We study phase transitions of a **4D extremal charged black hole** in General Relativity, when it is coupled to a Stückelberg scalar field.

SUMMARY OF THE FORMALISM

- **Near-horizon geometry of the extremal black hole in 4D with Λ has topology** $\mathrm{AdS}_2 \otimes \Sigma_k$

$$\boxed{ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu = v_1 \left(-r^2\, dt^2 + \frac{dr^2}{r^2} \right) + v_2\, d\Omega^2_{(k)}}$$

r = radial distance from the horizon
v_1 = radius of 2D anti-de Sitter space AdS_2
v_2 = radius of the transversal section Σ_k with the metric $\gamma_{nm}(y)$

$$d\Omega^2_{(k)} = \begin{cases} d\theta^2 + \sin^2\theta\, d\varphi^2\,, & y^m = (\theta, \varphi)\,, \quad k = 1 \\ dx^2 + dy^2\,, & y^m = (x, y)\,, \quad k = 0 \\ d\chi^2 + \sinh^2\chi\, d\varphi^2\,, & y^m = (\chi, \varphi)\,, \quad k = -1 \end{cases}$$

Note: $k = 0, -1$ is possible only when $\Lambda < 0$

Entropy function formalism

- **Action for gravity coupled to the EM and scalar fields**

  $I = \int d^4x \sqrt{-g}\, L(g, A, \phi)$

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  $$\boxed{H: \quad g_{\mu\nu} \to (v_1\,,\, v_2)\,, \quad A_\mu \to (e, p)\;, \quad \phi \to u}$$

  The scalar field, due to the attractor mechanism, depends only on it value on the horizon, u.

  The electromagnetic field on the horizon (based on isometries only) is $F_{rt} = e,\ F_{\theta\varphi} = \sqrt{\gamma}\,\frac{p}{4\pi}$

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- **Action evaluated on the horizon**

  $f(v, e, p, u) = \int_H d^2y \sqrt{-g}\, L(v, e, p, u)$

Entropy function formalism

- The function $f(v, e, p, u)$ satisfies the action principle – it has an extremum on the equations of motion, for given boundary conditions.
- **Boundary conditions**: Asymptotic electric charge q and magnetic charge p are kept fixed.

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Identification of asymptotic charges

- EM field equations and Bianchi identities lead to the first integrals
  $\int d^2y \, \frac{\delta I}{\delta F_{rt}} = a = const, \quad \int d^2y \, F_{\theta\varphi} = b = const.$
- Evaluated on the horizon, these integrals become $a = \frac{\partial f}{\partial e}$ and $b = p$
- Evaluated at the asymptotic infinity, these are the integrals of electric and magnetis fluxes, so that $b = p =$ magnetic charge and $a = q =$ electric charge

Entropy function formalism

- **Entropy function:** Legendre transformation of the function f with respect to the electric field $$E(v, e, p, u) = 2\pi \left[eq - f(v, e, p, u) \right]$$
- **Parameters near the horizon**: calculated as an extremum of the entropy function [Sen 2005] $$\frac{\partial E}{\partial v_i} = 0 \,, \quad \frac{\partial E}{\partial u} = 0 \,, \quad \frac{\partial E}{\partial e} = 0 \,, \quad \frac{\partial E}{\partial p} = 0$$

  e.g., $\frac{\partial E}{\partial e} = 0$ is equal to $q = \frac{\partial f}{\partial e}$
- **Black hole entropy:** extremum of the entropy function $$S = E_{\mathrm{extr}}$$
- Therefore, finding the entropy function $E(v, e, p, u)$ and its maximum, one can calculate the entropy, electric field and AdS_2 and Σ_k radii of the extremal black hole, independently on a particular solution considered.

Charged black hole coupled to a scalar field

THE ACTION

- **General Relativity with Λ + Maxwell field + complex scalar field**

$$I=\int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}\left(R-2\Lambda\right)-\frac{1}{4}F^2+L_S(\hat{\phi})\right]$$

- Complex scalar field $\hat{\phi}=\phi\,\mathrm{e}^{i\sigma}$

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- **Minimal coupling**
$L_S(\hat{\phi})=-\frac{1}{2}\,\left|(\partial-iA)\,\hat{\phi}\right|^2=-\frac{1}{2}\,\left[(\partial\phi)^2+\phi^2(\partial\sigma-A)^2\right]$
- **Local** $U(1)$ **symmetry**: $\phi\to\phi\,,\ \sigma\to\sigma+\alpha\,,\ A_\mu\to A_\mu+\partial_\mu\alpha$

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- **Non-minimal coupling with** $U(1)$ **symmetry:**
by the replacement $\phi^2\to P(\phi)\geq 0$

Charged black hole coupled to a scalar field

- **Stückelberg complex scalar**

$$L_S = -\frac{1}{2}\left[(\partial\phi)^2 + m^2\phi^2 + P(\phi)(\partial\sigma - A)^2\right]$$

- **Non-linear self-interaction choice**

$$P(\phi) = \phi^2 + \frac{a}{4}\phi^4 \geq 0$$

- When $P(\phi) = \phi^2$, the above action describes minimally coupled scalar field.
- Non-minimal coupling is introduced by the coupling constant $a \neq 0$.

Charged black hole coupled to a scalar field

- **Equations of motion**

$$
\begin{aligned}
&R_{\mu\nu} - \tfrac{1}{2}\, g_{\mu\nu}\, R + g_{\mu\nu}\,\Lambda = 8\pi G\, T_{\mu\nu} \\
&\nabla_\mu F^{\mu\nu} = P(\phi)\,(\nabla^\mu\sigma - A^\mu) \\
&(\Box - m^2)\phi = \tfrac{1}{2}\, P(\phi)\,(\nabla\sigma - A)^2 \\
&\nabla_\mu\,[P(\phi)\,(\nabla^\mu\sigma - A^\mu)] = 0 \;(\textit{not independent})
\end{aligned}
$$

- **Near-horizon parameters**

\- Field equation for $\sigma(x)$ is not independent due to the $U(1)$ symmetry and it can be gauge-fixed to $\sigma = 0$.

$\Rightarrow$ Extremal BH configurations are replaced by five parameters (v_1, v_2, e, p, u)

Charged black hole coupled to a scalar field

- **Lagrangian evaluated on the horizon**

$$L = \frac{1}{8\pi G}\left(\frac{k}{v_2} - \frac{1}{v_1} - \Lambda\right) + \frac{e^2}{2v_1^2} - \frac{p^2}{32\pi^2 v_2^2} - \frac{1}{2}\, m^2 u^2 + \frac{1}{2}\, P(u)\left(\frac{e^2}{v_1} - \frac{p^2 z_k(y)}{16\pi^2 v_2}\right)$$

- The function $z_k(y)$ depends on the horizon geometry and is given by $z_1 = \cot^2\theta$, $z_0 = x^2$ and $z_{-1} = \coth^2\chi$.

- **Transversal section volume element**

$$\mathrm{Vol}(\Sigma_k) = \int d^2y\,\sqrt{\gamma}$$

- **Auxiliary function**

$$f = \int d^2y\,\sqrt{\gamma}\, v_1 v_2\, L$$

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- Explicit dependence on y^m in the function $z_k(y)$ makes f divergent, unless the magnetic charge vanishes, $p = 0$.

- Magnetic field $F_{mn} \neq 0$ breaks a spherical symmetry of the black hole solution *in presence of the Stückelberg scalar*.

Charged black hole coupled to a scalar field

- **Charge density** $Q = q/\mathrm{Vol}(\Sigma_k)$
- **Entropy function**

$$E = 2\pi \mathrm{Vol}(\Sigma_k)\left[eQ - \frac{kv_1 - v_2 - \Lambda v_1 v_2}{8\pi G} - \frac{e^2 v_2}{2v_1} + \frac{v_1 v_2}{2}\left(m^2 u^2 - \frac{e^2}{v_1} P\right)\right]$$

- **Equations of motion** (extremum of E)

$$0 = \frac{\partial E}{\partial v_1} \quad \Rightarrow \quad k - \Lambda v_2 = v_2\left(\frac{e^2}{v_1^2} + m^2 u^2\right)$$

$$0 = \frac{\partial E}{\partial v_2} \quad \Rightarrow \quad 1 + \Lambda v_1 = \frac{e^2}{v_1} - v_1\, m^2 u^2 + e^2 P$$

$$0 = \frac{\partial E}{\partial e} \quad \Rightarrow \quad Q = v_2 e\left(\frac{1}{v_1} + P\right)$$

$$0 = \frac{\partial E}{\partial u} \quad \Rightarrow \quad 0 = 2v_1\, m^2 u - e^2 P'(u)$$

- **Convention** $4\pi G = 1$; **Choice** $e, Q > 0$ (no loss of generality)

Critical behavior

NORMAL PHASE

Reissner-Nordström (flat, dS,AdS) black hole

$u = 0$ it is always a particular solution of the scalar equation

- **General solution for fixed Q**

  $k \neq 0$ (*black holes with spherical and hyperbolic horizons*)

$$v_1^{(k)}(Q) = \frac{2Q^2}{1-4\Lambda Q^2 + k\sqrt{1-4\Lambda Q^2}},$$

$$v_2^{(k)}(Q) = \frac{2Q^2 k}{1+k\sqrt{1-4\Lambda Q^2}}$$

$$e^{(k)}(Q) = Q\,\frac{1+k\sqrt{1-4\Lambda Q^2}}{1-4\Lambda Q^2 + k\sqrt{1-4\Lambda Q^2}}$$

  $k = 0$ (*planar black holes or black branes*)

$$v_1^{(0)}(Q) = -\frac{1}{2\Lambda},$$

$$v_2^{(0)}(Q) = \frac{Q}{\sqrt{-\Lambda}}$$

$$e^{(0)}(Q) = \frac{1}{2\sqrt{-\Lambda}}$$

Critical behavior

- **Existence of the solution**

$\Lambda < 0$	There are 2 solutions with $k = \pm 1$
$0 < \Lambda < \frac{1}{4Q^2}$	There is 1 solution with $k = +1$
$\Lambda = 0$	Can be reproduced from the limit $\Lambda \to 0$ of the branch '$+$'
$\Lambda = \frac{1}{4Q^2}$	There is no finite solution for v_1

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- **Extremum of the entropy function**

$$S_k(Q) = \mathrm{Vol}(\Sigma_k)\,\frac{2\pi Q^2}{\sqrt{\Delta}+k}$$

$$S_0(Q) = \mathrm{Vol}(\Sigma_k)\,\frac{\pi Q}{\sqrt{-\Lambda}}\,, \qquad \Lambda < 0$$

- When $\Lambda = 0$, one gets the known result $S = q^2/4$ for the extremal Reissner-Nördstrom black hole.

Critical behavior

HAIRY PHASE

Hairy Reissner-Nordström (flat, dS,AdS) black hole

- Solution with the scalar hair exists only if $a \neq 0$ (nonlinear interaction), for the scalar masses $m \neq 0, 1, \frac{1}{2}$ and the cosmological constant $\Lambda \neq 0$
- **Three solutions for the scalar field** $u = 0\,, \qquad u = \pm\sqrt{\frac{2}{a}\left(\frac{v_1\, m^2}{e^2} - 1\right)}$
- When $u \neq 0$, the equations are invariant under the replacement $u \longrightarrow -u$, so we can chose $u > 0$.

$\Rightarrow$ this is known as the **basins of attraction**

Basins of attraction

- There is a **coexistence** of different well-defined solutions to the attractor equations stabilising the scalar field at the black hole horizon
- This suggests possible **phase transitions** by selecting different basins of attraction in the near-horizon dynamics of the scalar field
- Chosing $P(u)$ to be a polynomial of higher order, or even non polynomial, might lead to even larger basins of attraction – increase the number and complicate the structure
- Very little is known about basins of attraction in the literature, all of that in the SUGRA context, most of them associated to non-homogenous scalar manifolds, and none of the examples have $\Lambda \neq 0$

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Critical point

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$$v_{1c} = \frac{m^2-1}{\Lambda}\,, \qquad Q_c = \frac{k}{2m^2-1}\sqrt{\frac{m^2\left(m^2-1\right)}{\Lambda}}$$

$$v_{2c} = \frac{k\left(m^2-1\right)}{\Lambda(2m^2-1)}\,, \qquad e_c = \sqrt{\frac{m^2\left(m^2-1\right)}{\Lambda}}$$

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- The following inequalities must be fulfilled:

$$m^2 > 0, \qquad k\left(2m^2-1\right) > 0, \qquad \frac{m^2-1}{\Lambda} > 0$$

- **Critical parameters** are continuous for $k = \pm 1$
$v_i^{(k)}(Q_c) = v_{ic}\,, \;\; e^{(k)}(Q_c) = e_c$

- **Critical entropy** is continuous for $k = \pm 1$ $\boxed{S_c = S_k(Q_c)}$

Critical behavior

- **Planar horizons** give consistent eqs only for the particular scalar mass $m = \frac{1}{2}$, while the charge Q remains arbitrary.
- The critical value Q_c does not exist.
- Variations in the external parameter Q would not be able to induce an instability of the system and trigger a phase transition.
- We will not consider the black holes without Q_c.

Critical behavior

Classification of critical behavior

Standard classifications used in thermal field theories

- **Ehrenfest classification** is based on the Gibbs free energy G and it classifies transitions according to the order of the lowest derivative of $G(T)$ that shows a discontinuity upon crossing the coexistence curve. If there are divergences in discontinuities, Ehrenfest classification is not valid.

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- When $T \to 0$, it is more suitable to look at the classification based on the entropy $S(T)$.

- **Discontinued phase transitions** are the ones where S has a jump at the critical point. This is not our case.
- **Continued phase transitions** have S continuous. There exists the critical point where the response functions (i.e., heat capacity, compressibility, susceptibility, ...) possess discontinuities.

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- **Continued phase transitions** have S continuous. There exists the critical point where the response functions (i.e., heat capacity, compressibility, susceptibility, ...) possess discontinuities.

- In our case, we have to look at the entropy as a function of charge, $S(Q)$, and the response functions are its derivatives.

Critical behavior

Near-critical behaviour of the entropy

- We focus on $\Lambda \neq 0$ with the **spherical and hyperbolic horizons**, $k \neq 0$, and we know that $S(Q)$ is a continuous function of the charge at Q_c
- **Small parameter** $\epsilon = Q - Q_c$, which can be positive or negative

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- **Small parameter** $\epsilon = Q - Q_c$, which can be positive or negative
- **Critical exponents** $\beta, \delta, \alpha, \gamma$ use the power-law to describe how the parameters tend to the critical values:

$$\begin{aligned} u^2 &= A\,\epsilon^\beta + \cdots \\ e &= e_c + B\,\epsilon^\delta + \cdots \\ v_1 &= v_{1c} + V\,\epsilon^\alpha + \cdots \\ v_2 &= v_{2c} + C\,\epsilon^\gamma + \cdots \end{aligned}$$

- Solving the equations of motion in the leading order near Q_c gives rise to the *universal* critical exponents for any $k \neq 0$ $\boxed{\alpha = \beta = \gamma = \delta = 1}$
- This means that the scalar field behaves at the critical black hole horizon as $u = \sqrt{A(Q - Q_{c})} + \cdots$
- The sign of A detrmines for which Q (above or below Q_c) the parameter u is real

Solution of the fields equations when $Q = Q_c + \epsilon$

- **Behavior of the fields on the horizon**

$$u = \sqrt{A\epsilon} + \sqrt{\tilde{A}\epsilon}^3 + \cdots$$

$$v_1 = v_{1c} + B\,\epsilon + \tilde{B}\,\epsilon^2 + \cdots$$

$$v_2 = v_{2c} + C\,\epsilon + \tilde{C}\,\epsilon^2 + \cdots$$

$$e = e_c + D\,\epsilon + \tilde{D}\,\epsilon^2 + \cdots$$

- **Coefficients (unique)**

$$A = -\sqrt{\frac{m^2-1}{\Lambda m^2}}\,\frac{4k\Lambda^2(2m^2-1)^2}{\Lambda a-4(m^2-1)^3}$$

$$B = \frac{k\Lambda a\,(2m^2-1)^3}{\Lambda a-4(m^2-1)^3}$$

$$C = \sqrt{\frac{m^2(m^2-1)}{\Lambda}}\,\frac{2\Lambda a+4(m^2-1)^2}{\Lambda a-4(m^2-1)^3}$$

$$D = \sqrt{\frac{m^2(m^2-1)}{\Lambda}}\,\frac{2k\Lambda a\,(2m^2-1)^2}{\Lambda a-4(m^2-1)^3}$$

$$\vdots$$

Critical behavior

- **Two different solutions that extremize the entropy:**

- $u = 0$ for any Q
- $u = \sqrt{A\,(Q - Q_c)} + \cdots$ when $\mathrm{sgn}(A)\,(Q - Q_c) \geq 0$

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- **Calculation of the entropy near the critical point**

- When $\mathrm{sgn}(A)\left(Q - Q_c\right) > 0$, non-linear equations can be solved approximately in ϵ.
- When $\mathrm{sgn}(A)\left(Q - Q_c\right) < 0$, then $u = 0$ and thus $S_k(Q)$ is known exactly. This result can be compared to the previous one using the Teylor expansion in ϵ for $Q = Q_c + \epsilon$.

Critical behavior

- **Two different solutions that extremize the entropy:**

- $u = 0$ for any Q
- $u = \sqrt{A\,(Q - Q_c)} + \cdots$ when $\mathrm{sgn}(A)\,(Q - Q_c) \geq 0$

- **Calculation of the entropy near the critical point**

- When $\mathrm{sgn}(A)\,(Q - Q_c) > 0$, non-linear equations can be solved approximately in ϵ.
- When $\mathrm{sgn}(A)\,(Q - Q_c) < 0$, then $u = 0$ and thus $S_k(Q)$ is known exactly. This result can be compared to the previous one using the Teylor expansion in ϵ for $Q = Q_c + \epsilon$.

- **Entropy**

$$S|_{u\neq 0} = S_c + 2\pi^2 \mathrm{Vol}(\Sigma_k) e_c \epsilon + k\pi \mathrm{Vol}(\Sigma_k)(2m^2 - 1)^3\, \omega\, \epsilon^2 + O(\epsilon^3)$$
$$S|_{u=0} = S_c + S_c + 2\pi^2 \mathrm{Vol}(\Sigma_k) e_c \epsilon + k\pi \mathrm{Vol}(\Sigma_k)(2m^2 - 1)^3\, \epsilon^2 + \mathcal{O}(\epsilon^3)$$

where $\omega = \frac{\Lambda a}{\Lambda a - 4(m^2-1)^3}$

- The solution $u = 0$, that always exists, will change to the solution with $u \neq 0$ only if it has higher entropy, *i.e.*, $\omega > 1$

Critical behavior

- **Second Law of Thermodynamics**

  $\Delta S = S|_{u\neq 0} - S|_{u=0} > 0$

  It gives that a phase transition will happen if the hairy phase has larger entropy, that is, $\omega > 1$

- Shown behavior is very similar to the one described by the **Landau-Ginzburg phase transition theory**, only the free energy is replaced by the extremal black entropy and the temperature is replaced by the electric charge..

- **Discontinuity of the response function S'' typical for phase transitions:**

$$\boxed{S''_{<}(Q_c) \neq S''_{>}(Q_c)}$$

Critical behavior

When this transition happens?

- An interval of allowed scalar masses, for given Λ and k:

$$\begin{array}{llll} (a) & \Lambda > 0 & k = 1 & m^2 > 1 \\ (b) & \Lambda < 0 & k = 1 & \frac{1}{2} < m^2 < 1 \\ (c) & \Lambda < 0 & k = -1 & 0 < m^2 < \frac{1}{2} \end{array}$$

- Second Law of Thermodynamics of black holes:

$$\Lambda \left[\Lambda a - 4(m^2 - 1)^3 \right] > 0$$

- Existence of the hairy solution: $\operatorname{sgn}(A)(Q - Q_c) > 0$

$$\left[\Lambda a - 4(m^2 - 1)^3 \right] (Q - Q_c) < 0$$

- The bounds determine the coupling a

$$\boxed{a > \frac{4\left|m^2 - 1\right|^3}{|\Lambda|} > 0}$$

- Small or negative interactions do not favor the scalar hair.

Critical behavior

How the story goes?

- (a,c) or $\Lambda k > 0$: The RN (A)dS black hole exists for large charges. As the charge decreases and passes through the critical point, for $Q \leq Q_c$, the hairy solution appears, which has larger entropy.
- (b) or $\Lambda k > 0$: The RN AdS black hole is favored for small charges. As the charge increases, the hair grows when the charge crosses the critical point, $Q \geq Q_c$, also increasing the entropy of the configuration.

In all cases, there are phase transitions for either sign of Λ and any geometry of the horizon, provided the scalar coupling is strong enough.

EXAMPLES

- Let us choose $m^2 = \frac{3}{4}\,, \quad a = \frac{1}{12}\,, \quad \Lambda = -\frac{3}{\ell^2}\,, \quad \ell = 1$
- This choice fullfils $P(u) = u^2 + \frac{1}{12}\, u^4 > 0$
- Critical values of the parameters are $e_c = \frac{1}{4} \qquad Q_c = \frac{1}{2}$
  $v_{1c} = \frac{1}{12} \qquad v_{2c} = \frac{1}{6}$
- Near-critical solution for the scalar field,
  $u = 4\sqrt{Q - Q_c} + \frac{8}{3}\sqrt{\frac{(Q-Q_c)^3}{3}} + \mathcal{O}(\epsilon^{5/2})$ exists only if $Q \geq Q_c$, and $u = 0$ exists for any Q
- **Near-critical entropy**

$$S(Q) = \begin{cases} \frac{2\pi^2}{3} + 2\pi^2\,(Q - Q_c) + \frac{2\pi^2}{3}\,(Q - Q_c)^2 + \cdots\,, & Q \geq Q_c\,, \\ \frac{2\pi^2}{3} + 2\pi^2\,(Q - Q_c) + \frac{\pi^2}{2}\,(Q - Q_c)^2 + \cdots\,, & Q \leq Q_c\,. \end{cases}$$

- It is clear that the extremal black hole, under given conditions, indeed changes its phase at Q_c

Conclusions

- **Extremal AdS_4 black holes** with spherical and hyperbolic horizons can develop hair above/below some Q_c, due to variations of electric charge
- The mass of the Stückelberg scalar has to be in a given interval and the Stückelberg interaction non-linear and strong enough
- The phase transition does not occur when $\Lambda = 0$ or $k = 0$
- The scalar field plays the role of the order parameter
- Our calculations are done **analytically using the entropy function formalism**

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T H A N K   Y O U !