Some Geometrical Aspects of NC $SO(2,3)_*$ Gravity

Biljana Nikolić

Faculty of Physics, University of Belgrade, Serbia

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Outline

References

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Introduction

- General relativity and quantum field theory give good description of nature in their domains. There is a need for theory of gravity on quantum scales.
- One approach to this problem is introducing noncommutative spacetime

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}(\hat{x}).$$

$SO(2,3)$ gauge theory

- Generators of the $SO(2,3)$ group are M_{AB}, where A, B take values $0, 1, 2, 3, 5$.
- Commutation relations:

$$[M_{AB}, M_{CD}] = i(\eta_{AD}M_{BC} + \eta_{BC}M_{AD} - \eta_{AC}M_{BD} - \eta_{BD}M_{AC}),$$

where $\eta_{AB} = \mathrm{diag}(+,-,-,-,+)$ is $5D$ metric.

Representation of M_{AB}

- Generators of Clifford algebra in $5D$ are Γ_A, and they satisfy $\{\Gamma_A, \Gamma_B\} = 2\eta_{AB}$.
- The gamma matrices in $5D$ are $\Gamma_A = (i\gamma_a\gamma_5, \gamma_5)$, where γ_a, $(a = 0, 1, 2, 3)$ are the four-dimensional gamma matirces and γ_5 is defined by $\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.
- We can define generators M_{AB} using Γ_A as $M_{AB} = \frac{i}{2}[\Gamma_A, \Gamma_B]$.
- In this representation we obtain:

$$M_{ab} = \frac{i}{4}[\gamma_a, \gamma_b] = \frac{1}{2}\sigma_{ab} \; , \; M_{5a} = \frac{i}{2}\gamma_a.$$

The gauge potential

- $SO(2,3)$ gauge potential is

$$\omega_\mu = \frac{1}{2}\omega_\mu^{AB} M_{AB} = \frac{1}{4}\omega_\mu^{ab}\sigma^{ab} - \frac{1}{2}\omega_\mu^{a5}\gamma_a .$$

- Under infinitesimal gauge transformations ω_μ transforms as

$$\delta_\epsilon \omega_\mu = \partial_\mu \epsilon + i[\epsilon, \omega_\mu],$$

  where $\epsilon = \frac{1}{2}\epsilon^{AB} M_{AB}$ is gauge parameter.
- We identify ω_μ^{ab} with spin connection and $l\omega_\mu^{a5} = e_\mu^a$ with vierbeins, where l is some parameter which has dimension of lenght.

The field strength

- From gauge potential ω_μ the field strength is defined in the standard way:

$$F_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu - i[\omega_\mu, \omega_\nu] = \frac{1}{2}F_{\mu\nu}^{AB}M_{AB}.$$

- Under infinitesimal gauge transformations field strength transforms as

$$\delta_\epsilon F_{\mu\nu} = i[\epsilon, F_{\mu\nu}].$$

The field strength

- Components of the field strength $F^{AB}_{\mu\nu}$ decompse into $F^{ab}_{\mu\nu}$ and $F^{a5}_{\mu\nu}$ as

$$F^{ab}_{\mu\nu} = R^{ab}_{\mu\nu} - \frac{1}{l^2}(e^a_\mu e^b_\nu - e^b_\mu e^a_\nu),\ lF^{a5}_{\mu\nu} = \nabla_\mu e^a_\nu - \nabla_\nu e^a_\mu = T^a_{\mu\nu},$$

  where $R^{ab}_{\mu\nu} = \partial_\mu \omega^{ab}_\nu - \partial_\nu \omega^{ab}_\mu + \omega^{ac}_\mu \omega^{cb}_\nu - \omega^{bc}_\mu \omega^{ca}_\nu$ and $\nabla_\mu e^a_\nu = \partial_\mu e^a_\nu + \omega^{ab}_\mu e_{\mu b}$.
- $R^{ab}_{\mu\nu}$ is identified with the curvature tensor, and $T^a_{\mu\nu}$ with the torsion.

Gauge invariant actions

- We introduce an auxiliary scalar field $\phi = \phi^A \Gamma_A$ which transforms in the adjoint representation of the $SO(2,3)$ group $\delta_\epsilon \phi = i[\epsilon, \phi]$.
- Gauge invariant action is $S = c_1 S_1 + c_2 S_2 + c_3 S_3$ where

$$S_1 = \frac{il}{64\pi G_N} \int \epsilon^{\mu\nu\rho\sigma} \mathrm{Tr}(F_{\mu\nu} F_{\rho\sigma} \phi)$$

$$S_2 = \frac{1}{128\pi G_N l} \mathrm{Tr} \int d^4x \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} D_\rho \phi D_\sigma \phi \phi + c.c$$

$$S_3 = -\frac{i}{128\pi G_N l} \mathrm{Tr} \int d^4x \epsilon^{\mu\nu\rho\sigma} D_\mu \phi D_\nu \phi D_\rho \phi D_\sigma \phi \phi,$$

with $D_\mu \phi = \partial_\mu \phi - i[\omega_\mu, \phi]$.

Breaking of the $SO(2,3)$ gauge symmetry

- By choosing $\phi^a = 0$ and $\phi^5 = l$, we brake $SO(2,3)$ gauge symmetry down to $SO(1,3)$, and actions reduce to

$$S_1 = -\frac{1}{16\pi G_N}\int \mathrm{d}^4x\Big(\frac{l^2}{16}\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}R_{\mu\nu}^{\ ab}R_{\rho\sigma}^{\ cd} + \sqrt{-g}(R - \frac{6}{l^2})\Big)$$

$$S_2 = -\frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\Big(R - \frac{12}{l^2}\Big)$$

$$S_3 = -\frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\Big(-\frac{12}{l^2}\Big)$$

- We partially fix constants c_1, c_2 and c_3 by the requirement that the full action after the symmetry breaking reduces to the Einstein-Hilbert action with cosmological constant: $c_1 + c_2 = 1$ and $\Lambda = -3\frac{1+c_2+2c_3}{l^2}$.

NC space

- We work in the canonical (Moyal-Weyl, $\theta = const.$) NC spacetime.
- Following the deformation quantization approach, we represent noncommutative functions as functions of commutative coordinates and change the algebra multiplication with the Moyal-Weyl $\star$-product:

$$f(x) \star g(x) = e^{\frac{i}{2}\,\theta^{\mu\nu}\,\frac{\partial}{\partial x^\mu}\,\frac{\partial}{\partial y^\nu}} f(x)g(y)|_{y \to x},$$

  where $\theta^{\mu\nu}$ is a constant antisymmetric matrix and its components are small deformation parameters.
- For coordinates we obtain $[x^\mu \stackrel{\star}{,} x^\nu] = i\theta^{\mu\nu}$.

Noncommutative gauge theory

- In commutative case commutator of two infinitesimal gauge transformations is an infinitesimal gauge transformation: $[\delta_\alpha, \delta_\beta] = \delta_{-i[\alpha,\beta]}$, so we demand in NC case: $[\delta^*_\alpha \stackrel{\star}{,} \delta^*_\beta] = \delta^\star_{-i[\alpha,\beta]}$.
- Under NC infinitesimal gague transformations the gauge field, the field strength, auxiliary field transform as:

$$\delta^\star_\epsilon \hat{\omega}_\mu = \partial_\mu \hat{\Lambda}_\epsilon + i[\hat{\Lambda}_\epsilon \stackrel{\star}{,} \hat{\omega}_\mu], \quad \delta^\star_\epsilon \hat{F}_{\mu\nu} = i[\hat{\Lambda}_\epsilon \stackrel{\star}{,} \hat{F}_{\mu\nu}], \quad \delta^\star_\epsilon \hat{\Phi} = i[\hat{\Lambda}_\epsilon \stackrel{\star}{,} \hat{\Phi}].$$

- The quantities with hats are NC functions. They belong to enveloping algebra, because $\star$-commutators contain both commutators and anticommutators of generators of the gauge group. It seems that NC theory has infinitely many degrees of freedom.

The Seiberg-Witten map

- All NC fields and NC gauge parameter are functions only of the commutative fields and gauge parameter and their derivatives.
- NC gauge transformations are induced by the corresponding commutative gauge transformations.
- We can expand all NC quantities into power series in the noncommutative parameter $\theta^{\alpha\beta}$.

The Seiberg-Witten map solutions

- Expansions of the NC variables are:

$$\hat{\Lambda}_\epsilon = \epsilon + \hat{\Lambda}^{(1)} + \hat{\Lambda}^{(2)} + \dots, \quad \hat{\omega}_\mu = \omega_\mu + \hat{\omega}_\mu^{(1)} + \hat{\omega}_\mu^{(2)} + \dots \ ,$$

$$\hat{F}_{\mu\nu} = F_{\mu\nu} + \hat{F}_{\mu\nu}^{(1)} + \hat{F}_{\mu\nu}^{(2)} + \dots, \quad \hat{\Phi} = \Phi + \hat{\Phi}^{(1)} + \hat{\Phi}^{(2)} \dots$$

- Inserting those expansions into equation for closure of the NC gauge transformations and into transformation laws for NC fields, equations are solved order by order in noncommutative parameter.

The Seiberg-Witten map solutions

- The zeroth order solutions are commutative functions, and the first order is given by:

$$\hat{\Lambda}_\epsilon(1) = -\frac{1}{4}\theta^{\alpha\beta}\{\omega_\alpha, \partial_\beta\epsilon\},$$

$$\hat{\omega}_\mu^{(1)} = -\frac{1}{4}\theta^{\alpha\beta}\{\omega_\alpha, (\partial_\beta\omega_\mu + F_{\beta\mu}\},$$

$$\hat{F}_{\mu\nu}^{(1)} = -\frac{1}{4}\theta^{\alpha\beta}\{\omega_\alpha, \partial_\beta F_{\mu\nu} + D_\beta F_{\mu\nu}\} + \frac{1}{2}\theta^{\alpha\beta}\{F_{\mu\alpha}, F_{\nu\beta}\},$$

$$\hat{\phi}^{(1)} = -\frac{1}{4}\theta^{\alpha\beta}\{\omega_\alpha, (\partial_\beta + D_\beta)\phi\}.$$

- Recursively one can obtain the higher order corrections.

NC Gravity action

- We generalize the $SO(2,3)$ invariant commutative action by replacing fields with their NC counterparts and multiplication with $\star$-multiplication.
$S_{NC} = c_1 S_{1NC} + c_2 S_{2NC} + c_3 S_{3NC}$

$$S_{1NC} = \frac{il}{64\pi G_N} \mathrm{Tr} \int \mathrm{d}^4 x \epsilon^{\mu\nu\rho\sigma} \hat{F}_{\mu\nu} \star \hat{F}_{\rho\sigma} \star \hat{\phi} \, ,$$

$$S_{2NC} = \frac{1}{64\pi G_N l} \mathrm{Tr} \int \mathrm{d}^4 x \epsilon^{\mu\nu\rho\sigma} \hat{\phi} \star \hat{F}_{\mu\nu} \star \hat{D}_\rho \hat{\phi} \star \hat{D}_\sigma \hat{\phi} + c.c. \, ,$$

$$S_{3NC} = -\frac{i}{128\pi G_N l} \mathrm{Tr} \int \mathrm{d}^4 x \, \varepsilon^{\mu\nu\rho\sigma} D_\mu \hat{\phi} \star D_\nu \hat{\phi} \star \hat{D}_\rho \hat{\phi} \star \hat{D}_\sigma \hat{\phi} \star \hat{\phi}.$$

The Seiberg Witten expansion of the NC action

- We expand this action up to the second order in the deformation parameter: $S_{NC} = S^{(0)} + S^{(1)} + S^{(2)} + \dots$.
- The first order correction in the deformation parameter vanishes: $S_{NC}^{(1)} = 0$.
- Obtained second-order action is invariant under SO(2,3) gauge transformations.
- After the symmetry breaking, by setting the auxiliary field to $\phi^a = 0, \ \phi^5 = l$, we arive to SO(1,3) gauge invariant action.

The low-energy NC corrections to the gravity action

- We are interested in the low energy expansion of the NC action, so we keep only the terms of the zeroth, the first and the second order in the derivatives of the vierbeins (linear in curvature, quadratic in torsion...):

$$
\begin{aligned}
S_{NC} = \; & S + \frac{1}{128\pi G_N l^4}\int \mathrm{d}^4x\, e\theta^{\alpha\beta}\theta^{\gamma\delta}\Big((-2+12c_2+38c_3)R_{\alpha\beta\gamma\delta} \\
& +(4-18c_2-44c_3)R_{\alpha\gamma\beta\delta} - (6+22c_2+36c_3)g_{\beta\delta}R_{\alpha\gamma} + \frac{6+28c_2+56c_3}{l^2}g_{\alpha\gamma}g_{\beta\delta} \\
& +(5-\frac{9}{2}c_2-7c_3)T^a_{\ \alpha\beta}T_{\gamma\delta a} + (-10+\frac{9}{2}c_2+14c_3)T^a_{\ \alpha\gamma}T_{\beta\delta a} + (3-3c_2-2c_3)T_{\alpha\beta\gamma}T_{\delta\ \mu}^{\ \mu} \\
& +(1+2c_2)T_{\alpha\beta\rho}T^{\rho}_{\ \gamma\delta} + 8T_{\alpha\gamma\delta}T_{\beta\ \mu}^{\ \mu} - (2c_2+4c_3)T_{\alpha\gamma\rho}T^{\rho}_{\ \delta\beta} \\
& +(2c_2+4c_3)g_{\beta\delta}T_{\gamma\ \sigma}^{\ \sigma}T_{\alpha\ \rho}^{\ \rho} - (2c_2+4c_3)T_{\alpha\rho\sigma}T_{\gamma}^{\ \sigma\rho}g_{\beta\delta} + (-2+4c_2+18c_3)T_{\alpha\beta\gamma}e^{\rho}_{a}\nabla_\delta e^a_\rho \\
& +(6-8c_2-8c_3)T_{\alpha\gamma\beta}e^{\rho}_{a}\nabla_\delta e^a_\rho + (2+4c_2+12c_3)T_{\alpha\ \gamma}^{\ \mu}e^a_\beta\nabla_\delta e^a_\mu - T_{\alpha\ \beta}^{\ \mu}e^a_\delta\nabla_\gamma e^a_\mu \\
& +(-6-8c_2-16c_3)T_{\delta\rho\beta}e^{\rho}_{a}\nabla_\alpha e^a_\gamma - (2c_2+4c_3)g_{\alpha\gamma}T_{\mu\ \beta}^{\ \mu}e^{\rho}_{a}\nabla_\delta e^a_\rho - (2c_2+4c_3)g_{\beta\delta}T_{\alpha\ \rho}^{\ \sigma}e^{\rho}_{a}\nabla_\gamma e^a_\sigma \\
& -(4+16c_2+32c_3)e^{\mu}_{a}e_{b\beta}\nabla_\gamma e^a_\alpha\nabla_\delta e^b_\mu + (4+12c_2+32c_3)e_{\delta a}e^{\mu}_{b}\nabla_\alpha e^a_\gamma\nabla_\beta e^b_\mu \\
& -(2+4c_2+8c_3)g_{\beta\delta}e^{\mu}_{a}e^{\nu}_{b}\nabla_\gamma e^a_\mu\nabla_\alpha e^b_\nu + (2+4c_2+8c_3)g_{\beta\delta}e^{\mu}_{a}e^{\rho}_{c}\nabla_\alpha e^a_\rho\nabla_\gamma e^c_\mu\Big).
\end{aligned}
$$

Properties of the NC gravity action

- In the zeroth order this action reduces to the Einstein-Hilbert action with cosmological constant.
- Local Lorentz symmetry is perserved, but the action breaks the diffeomorphism symmetry.

Equations of motion

- Variation with respect to the vierbeins:

$$R_{\alpha\gamma}{}^{cd} e^\gamma_d e^\alpha_a e^\mu_c - \frac{1}{2} e^\mu_a R + \frac{3}{l^2}(1 + c_2 + 2c_3) e^\mu_a = \tau_a{}^\mu = -\frac{8\pi G_N}{e} \frac{\delta S^{(2)}_{NC}}{\delta e^a_\mu}.$$

- Variation with respect to the spin connection:

$$T_{ac}{}^c e^\mu_b - T_{bc}{}^c e^\mu_a - T_{ab}{}^\mu = S_{ab}{}^\mu = -\frac{16\pi G_N}{e} \frac{\delta S^{(2)}_{NC}}{\delta \omega^{ab}_\mu}.$$

- We see that the noncommutativity is a source of curvature and torison: i.e. flat spacetime becomes curved as an effect of noncommutative corrections; also, a torsion-free solution could develop a nonzero torsion in the presence of noncommutativity.

NC corrections to Minkowski space

- Minkowski space-time is a solution of vacuum Einstein equations without the cosmological constant $(1 + c_2 + 2c_3 = 0)$.
- We want to find NC corrections to this solution, and assume metric in the form: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ is a small correction of the order θ^2.
- The first equation of motion becomes

$$\frac{1}{2}(\partial_\sigma\partial^\nu h^{\sigma\mu} + \partial_\sigma\partial^\mu h^{\sigma\nu} - \partial^\mu\partial^\nu h - \Box h^{\mu\nu}) - \frac{1}{2}\eta^{\mu\nu}(\partial_\alpha\partial_\beta h^{\alpha\beta} - \Box h)$$
$$= \frac{11}{4l^6}(2\eta_{\alpha\gamma}\theta^{\alpha\mu}\theta^{\gamma\nu} + \frac{1}{2}\eta_{\alpha\gamma}\eta_{\beta\delta}\eta^{\mu\nu}\theta^{\alpha\beta}\theta^{\gamma\delta}).$$

- The second equation gives no contribution, so the Minkowski spacetime remains torsion-free.

Solution to equation of motion

- Righthand side of the equations for $h_{\mu\nu}$ are constant, so these equations are solved by $h_{\mu\nu}$ quadratic in the coordinates.
- We find the solution of the form:

$$g_{00} = 1 - \frac{11}{2l^6}\theta^{0m}\theta^{0n}x^m x^n - \frac{11}{8l^6}\theta^{\alpha\beta}\theta_{\alpha\beta}r^2, \qquad g^{0i} = -\frac{11}{3l^6}\theta^{0m}\theta^{0n}x^m x^n,$$

$$g_{ij} = -\delta_{ij} - \frac{11}{6l^6}\theta^{im}\theta^{jn}x^m x^n + \frac{11}{24l^6}\delta^{ij}\theta^{\alpha\beta}\theta_{\alpha\beta}r^2 - \frac{11}{24l^6}\theta^{\alpha\beta}\theta_{\alpha\beta}x^i x^j.$$

- Scalar curvature of this solution is $R = -\frac{11}{l^6}\theta^{\alpha\beta}\theta^{\gamma\delta}\eta_{\alpha\gamma}\eta_{\beta\delta} = const.$. Curvature is induced by the noncommutativity.

Metric in Fermi normal coordinates

- The Reimann tensor for this solution can be calculated easily. A very interesting (and unexpected) observation follows: knowing the components of the Riemann tensor the components of the metric tensor can be written as

$$g_{00} = 1 - R_{0m0n}x^m x^n, \qquad g_{0i} = -\frac{2}{3}R_{0min}x^m x^n, \qquad g_{ij} = -\delta_{ij} - \frac{1}{3}R_{imjn}x^m x^n.$$

- This shows that the coordinates x^μ we started with, are Fermi normal coordinates - inertial coordinates of a local observer moving along some geodesic. They can be constructed in a small neighborhood along the geodesic (cylinder).

Diffeomorphism symmetry breaking and Fermi normal coordinates

- The measurements performed by the local observer moving along the geodesic are described in the Fermi normal coordinates. So, that is the observer that measures $\theta^{\mu\nu}$ to be constant! In any other reference frame, observers will measure $\theta^{\mu\nu}$ different from constant.
- The breaking of diffeomorphism invariance is now understood better: there is a preferred reference system defined by the Fermi normal coordinates and the NC parameter $\theta^{\mu\nu}$ is constant in that particular reference system. The breaking of the diffeomorphism symmetry is a consequence of fixing of the coordinate system.