Phase transitions on the truncated Heisenberg space

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Introduction and motivation: Renormalizability on non-commutative spaces

Back in the day...

- PROBLEM: UV divergences in QED
- SOLUTION 1: momentum/energy UV cutoff
- uncertainty relations: max momentum ⇒ min distance
- Lorenz-invariant coordinate quantization (Snyder '47)
- SOLUTION 2: renormalization
- but GR is not renormalizable
- field localization leads to black hole formation
- NC coordinates ⇒ min uncertainty ⇒ space-time quantization

Non-commutative geometry

- systematic study since the 90ies
- effective appearance: limits of string theory, quantum Hall effect
- PROBLEM: UV/IR mixing

$$\Gamma^{(1)}_{\text{planar}} = \frac{1}{3}\int \frac{d^d k}{(2\pi)^d}\frac{1}{k^2+m^2}$$

$$\Gamma^{(1)}_{\text{non-planar}} = \frac{1}{6}\int \frac{d^d k}{(2\pi)^d}\frac{\exp(ik_\mu\theta_{\mu\nu}p_\nu)}{k^2+m^2} = \frac{1}{96\pi^2}\left(\Lambda^2_{\text{eff}} - m^2 \ln\frac{\Lambda_{\text{eff}}}{m^2}\right)$$

$$\Lambda_{\text{eff}} = \frac{1}{\frac{1}{\Lambda^2} - p_\mu\theta^2_{\mu\nu}p_\nu}$$

GW model

- Grosse and Wulkenhaar ’03

$$S_{GW} = \int d^{2n}x \left(\frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi + \frac{1}{2}\Omega^2(\tilde{x}_\mu\phi) \star (\tilde{x}^\mu\phi) + \frac{m^2}{2}\phi \star \phi + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi \right)$$

$$\tilde{x}_\mu = 2(\theta^{-1})_{\mu\nu}x^\nu$$

- Moyal $\star$ product

$$(f_1 \star f_2)(x) = \exp\left(\frac{i}{2}\theta^{\mu\nu}\partial_\mu^{x_1}\partial_\nu^{x_2}\right) f_1(x_1)f_2(x_2)\Big|_{x_1=x_2=x}$$

- NC coordinates

$$[x_\mu \stackrel{\star}{,} x_\nu] = i\theta^{\mu\nu}$$

- its renormalizability first studied in the matrix representation

GW model

$$S_{GW} = \int d^{2n}x \left(\frac{1}{2}\partial_\mu \phi \star \partial^\mu \phi + \frac{1}{2}\Omega^2 (\tilde{x}_\mu \phi) \star (\tilde{x}^\mu \phi) + \frac{m^2}{2} \phi \star \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \right)$$

- GW model is (super)renormalizable (Disertori et al. ’07)
- harmonic potential grants renormalizability
- harmonic potential can be interpreted as curvature (Burić and Wohlgenannt ’10)
- gravitation $\implies$ renormalization $\implies$ QFT?
- still no renormalizable noncommutative gauge model

Heisenberg Algebra $\mathfrak{h}$

$$[X, Y] = i$$

- infinite dim representation, $X \to +$, $Y \to -$:

$$\frac{1}{\sqrt{2}} \begin{bmatrix} & +1 & & & \\ +1 & & \pm\sqrt{2} & & \\ & +\sqrt{2} & & \pm\sqrt{3} & \\ & & +\sqrt{3} & & \ddots \\ & & & \ddots & \end{bmatrix}$$

Truncated Heisenberg Algebra $\mathfrak{h}^{\text{tr}}$

$$[X, Y] = i\,(1 - Z)\,, \quad [X, Z] = i\,\{Y, Z\}\,, \quad [Y, Z] = -i\,\{X, Z\}$$

$$Z = \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & N \end{bmatrix}$$

Modified truncated Heisenberg Algebra $\mathfrak{h}^{\mathrm{tr}}$

$$[X, Y] = i\,(1 - Z)\,, \quad [X, Z] = i\,\{Y, Z\}\,, \quad [Y, Z] = -i\,\{X, Z\}$$

$$[X, Y] = i\epsilon\,(1 - Z)\,, \quad [X, Z] = i\epsilon\,\{Y, Z\}\,, \quad [Y, Z] = -i\epsilon\,\{X, Z\}$$

Modified truncated Heisenberg Algebra $\mathfrak{h}_\epsilon^{\text{tr}}$

$$[X,Y] = i\epsilon\,(1-Z)\,, \quad [X,Z] = i\epsilon\,\{Y,Z\}\,, \quad [Y,Z] = -i\epsilon\,\{X,Z\}$$

$$\text{commutative limit} \quad \xleftarrow{\epsilon\to 0} \quad \mathfrak{h}_\epsilon^{\text{tr}} \quad \xrightarrow{\epsilon\to 1} \quad \mathfrak{h}^{\text{tr}} \quad \xrightarrow[N\xrightarrow{w}\infty]{z\to 0} \quad \mathfrak{h}$$

- possible to define differential calculus on $\mathfrak{h}_\epsilon^{\text{tr}}$
- curvature:

$$R = \frac{15}{4}\,\mathbb{1} - 2Z - 4\left(X^2 + Y^2\right)$$

- momenta:

$$\epsilon P_1 = -Y, \qquad \epsilon P_2 = X, \qquad \epsilon P_3 = \frac{\mathbb{1}}{2} - Z$$

- introducing mass scale: $X \to \mu X, \ldots$

QFT on $\mathfrak{h}_\epsilon^{\text{tr}}$

- scalar field coupled to curvature $\xi R\phi^2$; yields renormalizable GW model in $Z \to 0$ limit
- M. Burić, M. Wohlgenannt, ***Geometry of the Grosse-Wulkenhaar Model***, JHEP 1003 (2010) 053, [arXiv:hep-th/0902.3408] (2010)

- spinor field coupled to torsion $\text{Tr}\,(\psi\bar{\psi})(T_\alpha\gamma^\alpha)(\theta^\beta\gamma_\beta)$; yields renormalizable Vignes-Tourneret model
- M. Burić, J. Madore, L. Nenadović, ***Spinors on a curved noncommutative space: coupling to torsion and the Gross-Neveu model***, [arXiv:hep-th/1502.00761v1] (2015)

- gauge field, $\mathfrak{h}_\epsilon^{\text{tr}}$ through derivatives; nonrenormalizable: $1/\square$ and $1/\square^2$ non-local divergent terms
- M. Burić, L. Nenadović, D. Prekrat, ***One-loop structure of the*** $U(1)$ ***gauge model on the truncated Heisenberg space***, Eur. Phys. J. C 76 (2016) no.12, 672 [arXiv:hep-th/1610.01429v1]

Matrix models on NC spaces

- phase structure:
  - B. Ydri, R. Ahlam, R. Khaled, ***Emergent fuzzy geometry and fuzzy physics in 4 dimensions*** (2016) [arXiv:hep-th/1607.08296]
  - B. Ydri, K. Ramda, A. Rouag, ***Phase diagrams of the multitrace quartic matrix models of noncommutative*** ϕ^4 , Phys. Rev. D 93, 065056 (2016) [arXiv:hep-th/1509.03726]
  - R. Delgadillo-Blando, D. O'Connor, ***Matrix geometries and Matrix Models***, JHEP 11 (2012) 057 [arXiv:hep-th/1203.6901]
  - B. Spisso, ***First numerical approach to a Grosse-Wulkenhaar model*** (2011) [arXiv:hep-th/1111.2951]
  - F. G. Flores, D. O'Connor, X. Martin, ***Simulating the scalar field on the fuzzy sphere***, Int. J. Mod. Phys. A 24 (2009) 3917-3944 [arXiv:hep-lat/0903.1986]
  - ...

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  - ...
- GW phase structure not sufficiently explored (Spisso '11; Ydri et al. '16)
- new striped phase $\longleftrightarrow$ UV/IR mixing ?

Phase structure of the scalar model on $\mathfrak{h}^{tr}$

Phase structure exploration

- scalar matrix model on $\mathfrak{h}^{tr}$:

$$S_{K+R+PP} = \operatorname{Tr}\left(c_K \cdot \Phi[P_\alpha, [P_\alpha, \Phi]] + c_R \cdot R\Phi^2 + c_2 \cdot \Phi^2 + c_4 \cdot \Phi^4\right)$$

- model parameters: $\mu\,(=1)$, c_K, c_R, c_2, c_4
- "thermodynamic" quantities:
    - "energy" $\langle S \rangle$
    - "heat capacity" $C = \left\langle S^2 \right\rangle - \langle S \rangle^2$
    - "magnetization" $M = \langle |\operatorname{Tr}\Phi| \rangle$
    - "susceptibility" $\chi = \left\langle M^2 \right\rangle - \langle M \rangle^2$
- random field characteristics:
    - eigenvalue probability distribution
    - trace probability distribution

Phase structure exploration

- scalar model:

$$S_{K+R+PP} = \mathrm{Tr}\left(c_K \cdot \Phi[P_\alpha,[P_\alpha,\Phi]] + c_R \cdot R\Phi^2 + c_2 \cdot \Phi^2 + c_4 \cdot \Phi^4\right)$$

- numerical treatment: Monte Carlo
- control: Schwinger-Dyson identity: $\langle 2S_2 + 4S_4 \rangle = N^2$
- analytical treatment: multitrace expansion

$$d\Phi = \prod_i d\Phi_{ii} \prod_{i<j} d\,\mathrm{Re}\{\Phi_{ij}\} d\,\mathrm{Im}\{\Phi_{ij}\} = dU \prod_i d\lambda_i\, \Delta_N^2(\lambda)$$

- Vandermonde determinant:

$$\Delta_N^2(\lambda) = \prod_{i<j} (\lambda_i - \lambda_j)^2$$

Phase structure

- EOM:

$$2c_K[P_\alpha, [P_\alpha, \Phi]] + c_R\{R, \Phi\} + 2\Phi\left(c_2 + 2c_4\Phi^2\right) = \mathbb{0};$$

- extrema:

$$\Phi_K = \frac{\mathrm{Tr}\Phi}{N}\,\mathbb{1}, \qquad \Phi_R = \mathbb{0}, \qquad \Phi_{PP}^2 = \begin{cases} \mathbb{0} & \text{for } c_2 \geq 0, \\ -\dfrac{c_2\,\mathbb{1}}{2c_4} & \text{for } c_2 < 0. \end{cases}$$

- disordered phase: $\Phi_{\updownarrow} = \mathbb{0}$
- non-uniformly ordered phase: $\Phi_{\uparrow\downarrow} \propto U\,\mathbb{1}_\pm\,U^\dagger$, $\sigma(\mathbb{1}_\pm) = \{\pm 1\}$
- uniformly ordered phase: $\Phi_{\uparrow\uparrow} \propto \mathbb{1}$

Phase structure

- EOM:

$$2c_K[P_\alpha,[P_\alpha,\Phi]] + c_R\{R,\Phi\} + 2\Phi\left(c_2 + 2c_4\Phi^2\right) = \mathbb{0};$$

- extrema:

$$\Phi_K = \frac{\mathrm{Tr}\Phi}{N}\,\mathbb{1}, \qquad \Phi_R = \mathbb{0}, \qquad \Phi_{PP}^2 = \begin{cases} \mathbb{0} & \text{for } c_2 \geq 0, \\ -\frac{c_2\,\mathbb{1}}{2c_4} & \text{for } c_2 < 0. \end{cases}$$

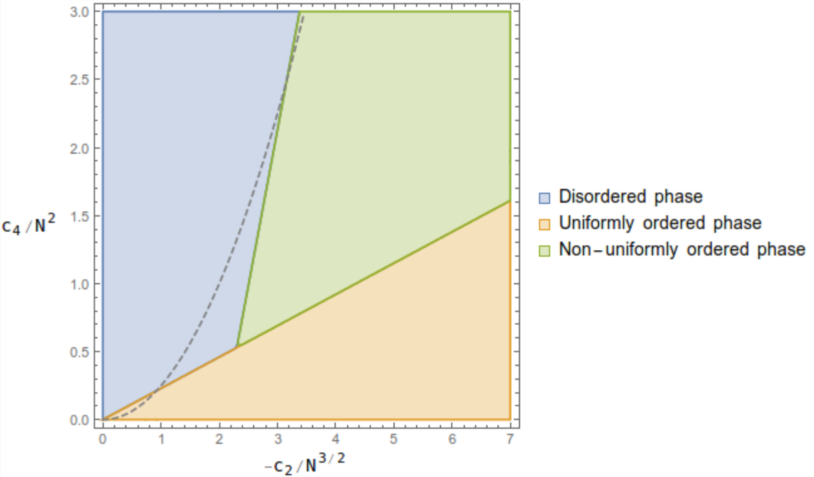
- disordered phase: $\Phi_{\updownarrow} = \mathbb{0}$
- non-uniformly ordered phase: $\Phi_{\uparrow\downarrow} \propto U\,\mathbb{1}_\pm\,U^\dagger$, $\sigma(\mathbb{1}_\pm) = \{\pm 1\}$
- uniformly ordered phase: $\Phi_{\uparrow\uparrow} \propto \mathbb{1}$

- modified ordered phases:

$$\Phi^2 = -\frac{c_2\,\mathbb{1} + c_R R}{2c_4}$$

- expulsion of the $\uparrow\downarrow$-phase?

Phase structure

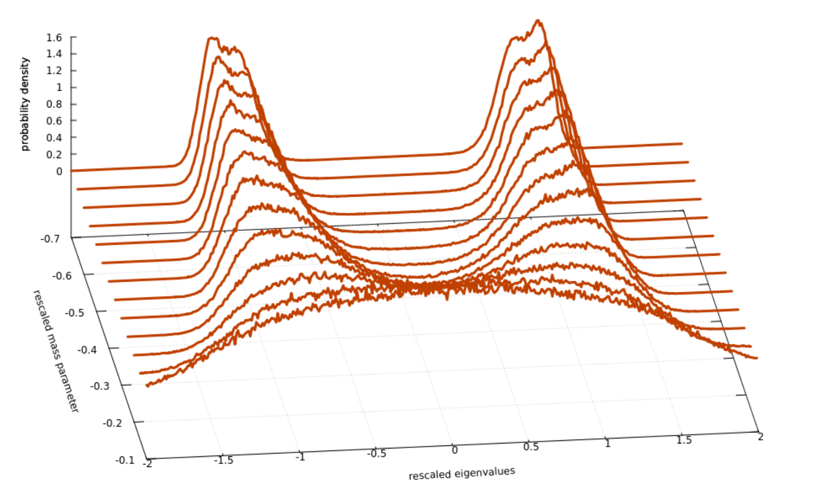


First results

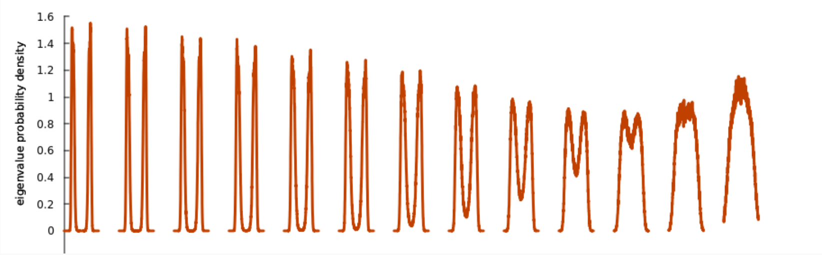
First results

- STSM @ DIAS (Dublin, Ireland), March ’17
- basic code in collaboration with D. O’Connor i S. Kováčik
- analytical results for trace distribution for $N = 2$ confirmed
- previous results for PP model confirmed (Flores et al. ’09)

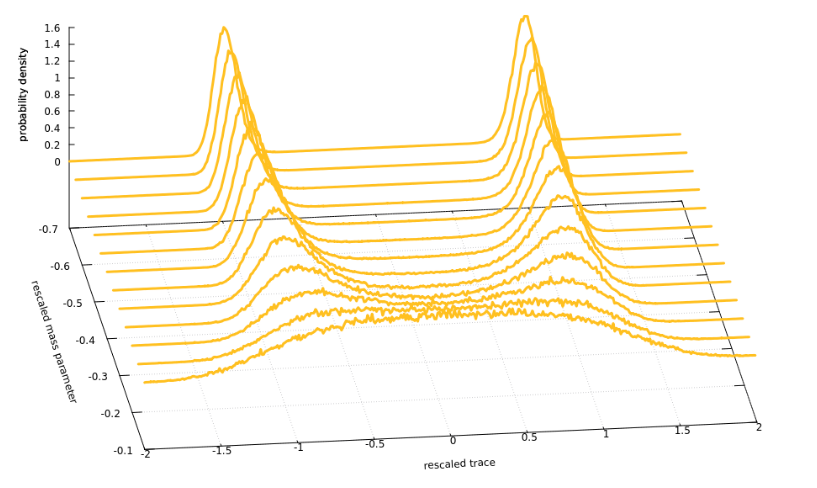
Eigenvalue probability distribution ($N = 2$)



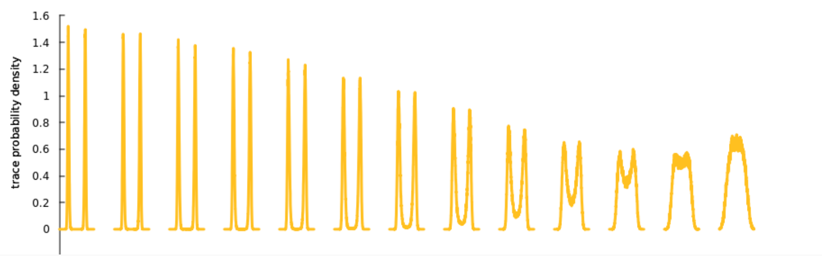
Eigenvalue probability distribution ($N = 2$)



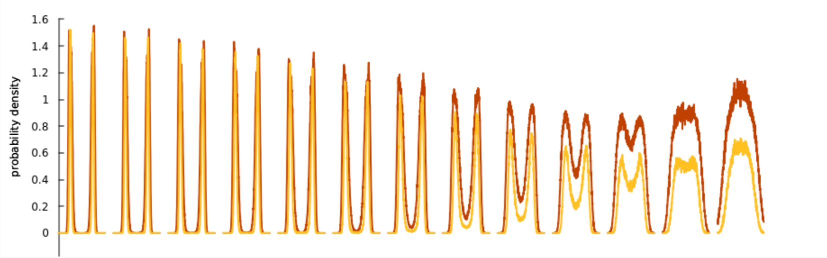
Trace probability distribution ($N = 2$)



Trace probability distribution ($N = 2$)



Comparison: Eigenvalue vs. trace distribution ($N = 2$)



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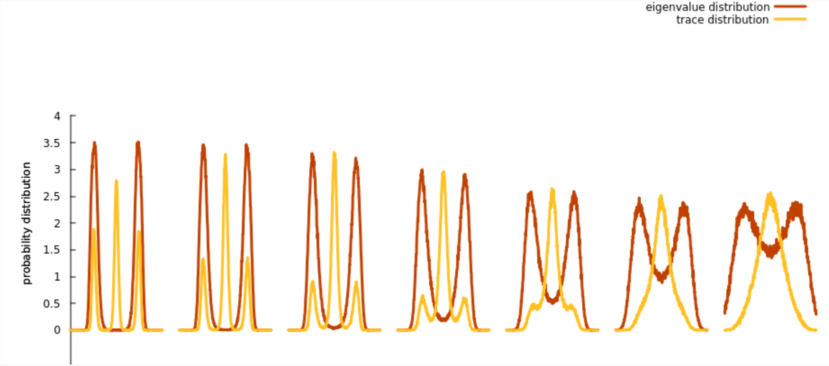

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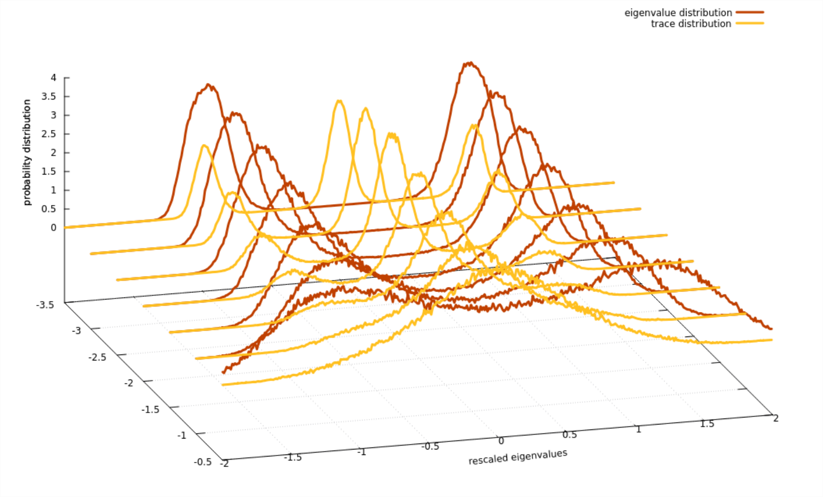
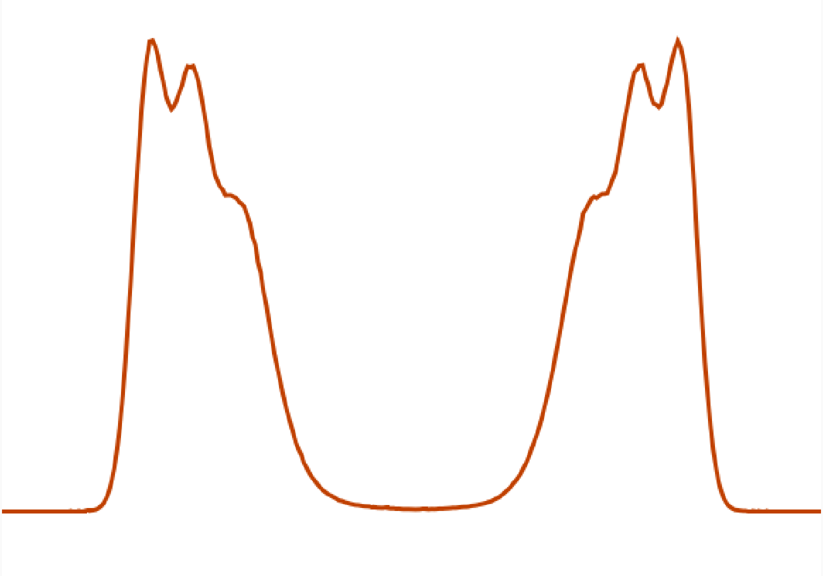
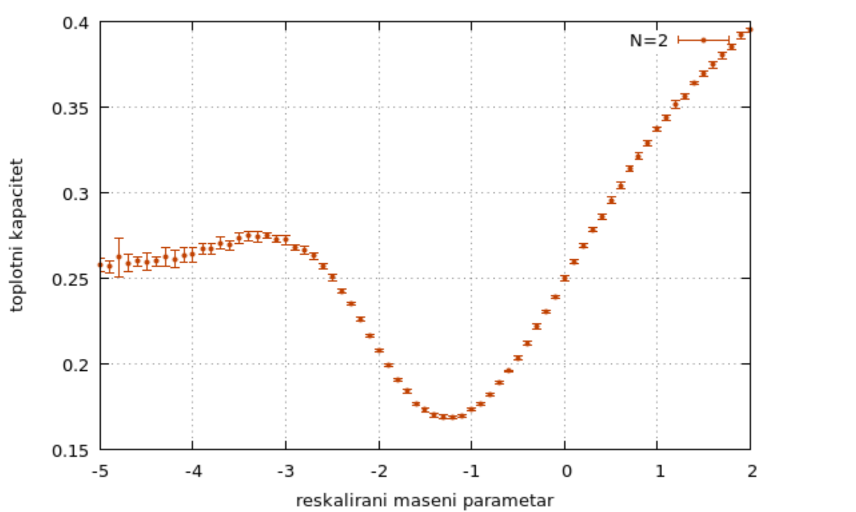


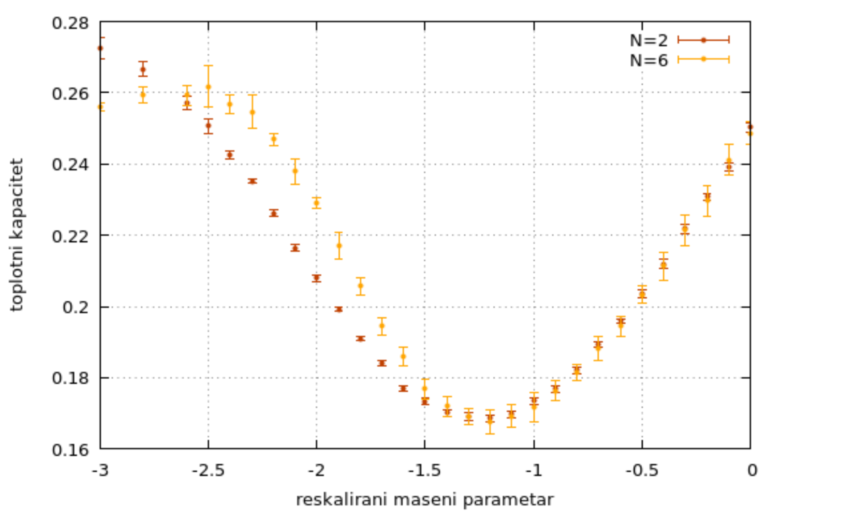
Repulsion of degenerate eigenvalues ($N = 6$)



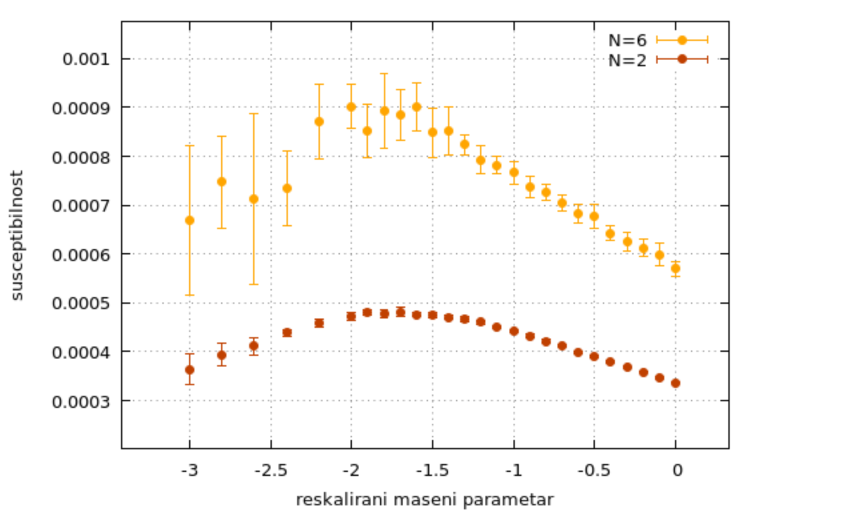
Heat capacity as a function of rescaled mass parameter



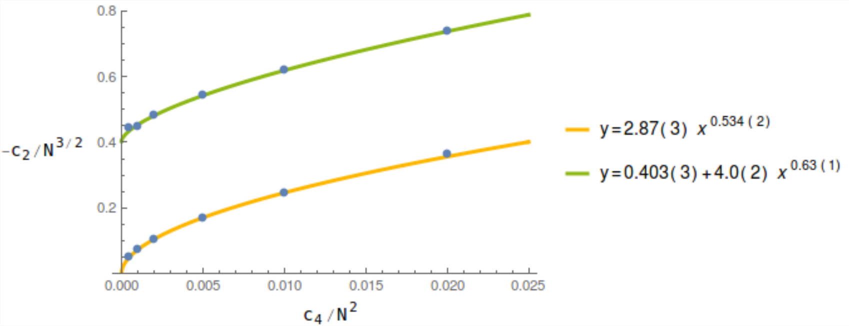
Heat capacity as a function of rescaled mass parameter (collapsed data)



Susceptibility as a function of rescaled mass parameter (collapsed data)



First results: Phase diagram & modified ordered phase



The End

Thank you for the attention.

Appendix: Fuzzy-sphere

- j-dim IR $SU(2)$: $[J_a, J_b] = i\epsilon_{abc} J_c$
- Casimir: $J_1^2 + J_2^2 + J_3^2 = j(j+1)$
- radius: R
- coordinates:

$$x_a = \frac{R}{\sqrt{j(j+1)}} J_a, \qquad x_1^2 + x_2^2 + x_3^2 = R^2$$

- integral:

$$\int f(x_a) := \frac{4\pi R^2}{\sqrt{j(j+1)}} \operatorname{Tr} F(x_a)$$

- surface area:

$$S = \int \mathbb{1} := \frac{4\pi R^2}{\sqrt{j(j+1)}} \cdot j = 4\pi R^2 \sqrt{\frac{j}{j+1}}$$