Randall-Sundrum Braneworld and Holography for 5D Chern-Simons Gravity

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Dimensionality of spacetime

Let's go back to 1747. Immanuel Kant published a paper titled "*Gedanken von der wahren Schätzung der lebendigen Kräfte*". In this paper, he suspected that the dimensionality of our space is determined from the inverse square law of Newtonian gravity $F = -G\frac{Mm}{r^2}$.

"Accordingly, I am of the opinion that substances in the existing world, of which we are a part, have essential forces of such a kind that they propagate their effects in union with each other according to the inverse relation of the distances; secondly, that the whole to which this gives rise has, by virtue of this law, the property of being three-dimensional; thirdly, that this law is arbitrary, and that God could have chosen another, e.g., the inverse-cube, relation; fourthly, and finally, that an extension with different properties and dimensions would also have resulted from a different law." [Kant]

Dimensionality of spacetime

Modern point of view: extra dimensions are possible (and in some theories inevitable), though the observed physics at observable length scales should be four dimensional. Theory often led us to consider higher dimensional spacetimes, in agreement with the previous quote. Therefore, we must find a way to obtain four-dimensional physics from those higher dimensional models. Original idea was presented a century ago by Kalutza and Klein, and uses a concept of KK compactification. [KK]

Another approach: Braneworld scenarios [ADD,RS,KR,...]

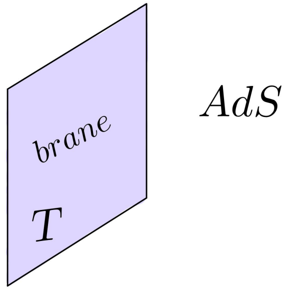
For phenomenological reasons, Randall and Sundrum introduced a model where the starting point was a five dimensional AdS spacetime with branes. [RS]

RS models

The initial motivation to cure the hierarchy problem: RS I model (two branes).
RS II model involves only one brane. Two sided version: analogue of a delta function potential in QM. [Randall, Sundrum, Karch,...]

$$\frac{1}{16\pi G}\int_N \mathrm{d}x^5\,\sqrt{-g}R - T\int_Q \mathrm{d}x^4\,\sqrt{-h} + GHY$$

One can also put matter fields on the brane.
See a beautiful review talk by A. Karch at Strings 2022
(BTW, this is a string theory inspired model and is not a product of string theory)



Chern-Simons gravity

Two (different) perspectives. Chern-Simons gravity as a gauge theory for an AdS Lie group, vs Lovelock Chern-Simons theory, as a special point in the space of Lovelock theories. Chern-Simons gravity is defined only in an odd number of dimensions. For phenomenology, we stick to $D = 5$ dimensions.
Motivation: relate this theory to some more phenomenologically acceptable [Zanelli, Izaurieta,Rodriguez,...]
A generalisation of Lovelock theorem allows us to consider the most general action (without explicit torsion) of the form

$$L = c_1 \varepsilon_{abcde} R^{ab} R^{cd} e^e + c_2 \varepsilon_{abcde} R^{ab} e^c e^d e^e + c_3 \varepsilon_{abcde} e^a e^b e^c e^d e^e.$$

In an odd number of dimensions, one can define Chern-Simons action as $\mathrm{d}L_{CD} \sim \langle F^{\frac{D+1}{2}} \rangle$, where F is a curvature two form for a given gauge connection.

$SO(4,2)$ Chern-Simons gravity

Choosing a gauge group $SO(4,2)$, and a gauge connection decomposition [Zanelli, Chamseddine]

$$A = \frac{1}{2}\hat{\omega}^{AB} J_{AB} + \frac{1}{l}\hat{e}^{A} P_{A},$$

with

$$[J_{AB}, J_{CD}] = G_{AD} J_{BC} + G_{BC} J_{AD} - (C \leftrightarrow D),$$
$$[J_{AB}, J_{C5}] = G_{BC} J_{A5} - G_{AC} J_{B5},$$
$$[J_{A5}, J_{C5}] = J_{AC}.$$

Chern Simons action $\int\limits_{\mathcal{N}} \langle F^2 A - \frac{1}{2} F A^3 + \frac{1}{5} A^5 \rangle$ gives

$$S_{CS} = k \int \varepsilon_{ABCDE} \left(\frac{1}{l} \hat{R}^{AB} \hat{R}^{CD} \hat{e}^{E} + \frac{2}{3l^3} \hat{R}^{AB} \hat{e}^{C} \hat{e}^{D} \hat{e}^{E} + \frac{1}{5l^5} \hat{e}^{A} \hat{e}^{B} \hat{e}^{C} \hat{e}^{D} \hat{e}^{E} \right).$$

Chern-Simons gravity

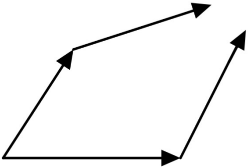
The last equality holds up to a boundary term. From now on, we consider a given Lagrangian that is obviously a particular choice of Lovelock Lagrangians. In a more familiar form

$$S_{CS} = \frac{1}{16\pi G}\int \mathrm{d}^5x\sqrt{-g}\Big[R - 2\Lambda \\ + \frac{l^2}{4}\left(R^2 - 4R^{\mu\nu}R_{\nu\mu} + R^{\mu\nu\rho\sigma}R_{\rho\sigma\mu\nu}\right)\Big]$$

We set $l = 1$ for simplicity. Varying with respect to independent fields $\hat{e}^A$ and $\hat{\omega}^{AB}$, we obtain equations of motion. However, note that the relative values of coefficients multiplying different terms are fixed by l, and that we cannon choose this parameter to obtain EH term as dominant (cf. effective gravity action in string theory). Equations of motion allows for a nonzero torsion. Also, AdS spacetime is a solution to those equations ($\hat{R}^{AB} + \hat{e}^2\hat{e}^B = 0$).

Torsion

General relativity was formulated on Riemannian manifolds, where torsion is set to zero. An alternative approach to classical GR would be to consider action $\int \varepsilon_{abcd} R^{ab} e^c e^d$ in the first order formalism, with e^a and ω^a independent, and then to set $T^a = 0$ through the equations of motion for a spin-connection. Nevertheless, it is an important question whether torsion is present in quantum mechanical description of gravity.



$$\int \mathcal{D}e\mathcal{D}\omega \; e^{iS}$$

In four dimensions, one possible solution would be the introduction of Holst-Barbero-Immirizi term. In three dimensions, in the context of AdS/BCFT, this is a work in progress. Also: fermions, supergravity.

RS setup

We will assume that the manifold is not (pseudo)Riemannian. This is the most important difference from previous considerations. Action is given by

$$S_{CS} + S_{brane}.$$

In the one-sided version, the brane is used as an IR cutoff, and integration is performed only over one part of the manifold. We then use holographic renormalization [Banados, Miskovic, Theisen, Cvetkovic, Simic,...]. Usual motivation: on-shell action suffers from the infrared divergences (and dual CFT has UV divergences) AdS/CFT prescription [Witten, Maldacena,...]

$$\delta S_{ren} = \int \tau^a \delta e_a + \frac{1}{2}\sigma^{ab}\delta\omega_{ab}.$$

Here, we have a cutoff CFT - a finite part that is not determined solely by the boundary fields. Its variations contains further ϵ finite terms.

Holographic renormalization

Using appropriate gauge choice [Banados, Miskovic, Theisen, Miskovic, Cvetkovic, Simic], Fefferman-Graham expansion is finite

$$\mathrm{d}s^2 = \frac{\mathrm{d}\rho^2}{4\rho^2} + \frac{1}{\rho}(g_{\alpha\beta} + 2\rho k_{(ab)} + \rho^2 k^a_\alpha k_{a\beta})\mathrm{d}x^\alpha \mathrm{d}x^\beta .$$

CS Lagrangian is gauge invariant up to boundary terms, so different asymptotic gauge conditions lead to different theories. Fields e^a, ω^{ab} and k^a are boundary fields. e^a and ω^{ab} are interpreted as sources in a dual CFT, while k^a is connected with one-point functions and is not determined from the CFT geometry. They have to satisfy identities coming from the bulk equations. They give rise to a holographic Ward identities in a dual language.

Tension

We place a brane at $\rho = \epsilon$. The tension term added to the brane Lagrangian is

$$S_{brane} = -T \int_{\mathcal{Q}} \varepsilon_{abcd} \hat{e}^a \hat{e}^b \hat{e}^c \hat{e}^d$$

Here, T is poportional to the brane tension. One could add matter fields confined to the brane, that would make the setup more realistic, but we stick to the simplest choice. Constraints that have to be satisfied are

$$\begin{aligned}
&\varepsilon_{abcd}(R^{ab} + 4e^a k^b)(R^{ab} + 4e^a k^b) = 0, \\
&\varepsilon_{abcd}(R^{bc} + 4e^b k^c)T^d = 0, \\
&\varepsilon_{abcd}(R^{bc} + 4e^c k^c)Dk^d = 0, \\
&\varepsilon_{abcd}\big((R^{cd} + 4e^c k^d)e^e k_e + 2T^c Dk^d\big) = 0,
\end{aligned}$$

They have to be introduced via Lagrange multipliers to make variations of boundary fields independent on the brane.

AdS/CFT and RS braneworld

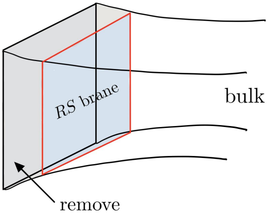
Equations of motion are obtained from

$$\delta\Big[W_{CFT} + 4k\int \varepsilon_{abcd}R^{ab}k^c e^d + \int \varepsilon_{abcd}\Big(\frac{2}{3}k - T\Big)e^a e^b e^c e^d$$
$$+ \int \varepsilon_{abcd}\Big(4k - 6T\Big)e^a k^b k^c k^d - 2k\int \varepsilon_{abcd}R^{ab}e^c e^d$$
$$- \int \varepsilon_{abcd}\Big(\frac{8}{3}k - 4T\Big)e^a e^b e^c k^d - T\int \varepsilon_{abcd}\Big(4e^a k^b k^c k^d + k^a k^b k^c k^d\Big)\Big].$$

Here, we consider "Minkowski" branes, tuning the brane tension to be $T = \frac{2}{3}k$. We recognise the EH term in the last expression (but note the minus sign).

Equations

Usually, induced metric $\gamma_{\alpha\beta}$ is used, but here we use the asymptotic boundary fields (rescaled such that they correspond to the physical fields induced on the brane). Also, we insist on *not* adding the GHY term, as e^a and ω^{ab} are independent. By putting $\epsilon \to 0$, we recover standard AdS/CFT correspondence. Keeping the boundary at a finite distance gives rise to the boundary dynamical gravity, that is not present in a dual filed theory in standard AdS/CFT. We used this relation to holography, as it is a good guiding principle.



Equations

$$
\begin{aligned}
& k\varepsilon_{abcd}\Big(-4R^{ab}k^c - 16e^a k^b k^c - 4R^{ab}e^c\Big) \\
& +\varepsilon_{abcd}\Big(8\phi_1(R^{ab}+4e^a k^b)k^c - 4\phi_2^a k^b T^c - D\phi_2^a(R^{bc}+4e^b k^c) - 4\phi_2^a T^b k^c + 4\phi_2^a e^b Dk^c - 4\phi_3^a k^b Dk^c \\
& \qquad +4\phi_4^{ab}k^c e^m k_m + 2D\phi_4^{ab}Dk^c + 2\phi_4^{ab}R^c{}_m k^m\Big) - \varepsilon_{abcm}\phi_4^{ab}(R^{cm}+4e^c k^m)k_d = 0,
\end{aligned}
$$

$$
\begin{aligned}
& k\varepsilon_{abcd}\Big(4R^{ab}e^c + 4R^{ab}k^c - \frac{16}{3}k^a k^b k^c\Big) \\
& +\varepsilon_{abcd}\Big(8\phi_1(R^{ab}+4e^a k^b)e^c - 4\phi_2^a e^b T^c - 4\phi_3^a e^b Dk^c - D\phi_3^a(R^{bc}+4e^b k^c) \\
& \qquad - 4\phi_3^a D(e^b k^c) - 2D(\phi_4^{ab}T^c) + 4\phi_4^{ab}e^c e^m k_m\Big) + \varepsilon_{abcm}\phi_4^{ab}(R^{cm}+4e^c k^m)e^d = 0,
\end{aligned}
$$

$$
\begin{aligned}
& -4\varepsilon_{abcd}\Big(Dk^c e^d + k^c T^d + e^c T^d + Dk^c k^d\Big) \\
& +\varepsilon_{abcd}\Big(-2D\phi_1(R^{cd}+4e^c k^d) - 8\phi_1 T^c k^d + 8\phi_1 e^c Dk^d D\phi_2^c T^d + \phi_2^c R^d{}_m e^m - D\phi_3^c Dk^d \\
& \qquad + \phi_3^c R^d{}_m k^m - D\phi_4^{cd}e^m k_m + \phi_4^{cd}T^m k_m - \phi_4^{cd}e^m Dk_m\Big) \\
& +\varepsilon_{amcd}\Big(\phi_2^c(R^{dm}+4e^d k^m)e_b + \phi_2^c(R^{dm}+4e^d k^m)k_b - 2\phi_4^{cd}Dk^m e_b + 2\phi_4^{cd}T^m k_b\Big) = 0.
\end{aligned}
$$

pp-waves

Solutions: Minkowski spacetime on the brane (trivial, but good for consistency), pp waves without torsion.

pp-waves correspond to a specific spacetime model designed to describe the propagation of different planar waves. We have

$$\mathrm{d}s^2 = 2H\mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}v + \mathrm{d}x^2 + \mathrm{d}y^2$$

Einstein equations: $\partial_x\partial_x H + \partial_y\partial_y H = 0$. Given spacetime also solves the constraint $\varepsilon_{abcd}R^{ab}R^{cd} = 0$.
In five dimensions, similar equations and conclusions have been obtained from a different perspective [Edelstein, Hassaıne, Troncoso, Zanelli]. This opens a possible connection between our model and those models seeking to obtain GR from CS gravity.

Conclusion and future directions

1. Braneworld scenario can be realised in the first order formalism, starting from the CS action.
2. Obtained equations are complicated but have some simple solutions.
3. It would be interesting to find more complicated solutions (Minkowski and pp waves).
4. One can consider de-Sitter branes. It is interesting to check whether dS is a solution of derived equations, and whether we can compute dS entropy as an entanglement entropy using RT prescription [Shiromizu, Izumi, Kushihara,...]
5. Work on AdS/BCFT.

Thank you for your attention!