Entropy of extremal black holes in Poincare gauge theory: the case of rotating black hole

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Quick overview of Poincare gauge theory: fields

- Basic variables:

Tetrad i spin connection	Torsion and curvature
$b^i = b^i{}_\mu dx^\mu$	$T^i = db^i + \omega^i{}_k \wedge b^k$
$\omega^{ij} = \omega^{ij}{}_\mu dx^\mu$	$R^{ij} = d\omega^{ij} + \omega^i{}_k \wedge \omega^{kj}$

Quick overview of Poincare gauge theory: gravitation Lagrangian and covariant momenta

- General form of gravitational Lagrangian in PGT

$$L_G = -^*(a_0 R + 2\Lambda) + T^i \sum_{n=1}^{3} {}^*(a_n\, {}^{(n)}T_i) + \frac{1}{2} R^{ij} \sum_{n=1}^{6} {}^*(b_n\, {}^{(n)}R_{ij})$$

- From the gravitational Lagrangian, the covariant momenta are calculated as:

$$H_i = \frac{\partial L_G}{\partial T^i} \qquad H_{ij} = \frac{\partial L_G}{\partial R^{ij}}$$

- Corresponding energy-momentum and spin currents:

$$E_i = \frac{\partial L_G}{\partial b^i} \qquad E_{ij} = \frac{\partial L_G}{\partial \omega^{ij}}$$

Quick overview of Poincare gauge theory: field equations

- From the previous, the fields equations of PGT can be written in compact form

$$\nabla H_i + E_i = 0 \qquad \nabla H_{ij} + E_{ij} = 0$$

Entropy of black holes in Poincare gauge theory: canonical generator

- The base of Hamiltonian approach to calculating entropy is based on the existence of canonical gauge generator G.
- G is given by an integral on a spacelike slice Σ, G acts on phase-space variables via Poisson bracket.
- G must have well defined functional derivatives, which is assured by adding surface correction terms Γ.
- The improved generator $\tilde{G} = G + \Gamma$ is regular.

Entropy of black holes in Poincare gauge theory: variation of the surface terms

- In the case of a black hole, the boundary surface has two distinct parts - one at infinity and one on the horizon
- Correspondingly the correction term Γ has two parts: $\Gamma = \Gamma_\infty + \Gamma_H$
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$$\delta\Gamma_\infty = \oint_{S_\infty} \delta B(\xi) \qquad \delta\Gamma_H = \oint_{S_H} \delta B(\xi)$$

$$\delta B(\xi) = (\xi \lrcorner b^i) H_i + \delta b^i (\xi \lrcorner H_i) + \frac{1}{2}(\xi \lrcorner \omega^{ij})\delta H_{ij} + \frac{1}{2}\delta\omega^{ij}(\xi \lrcorner H_{ij})$$

Entropy of black holes in Poincare gauge theory: conserved charges

- Γ_∞ are interpreted as asymptotic charges, if they're solutions of the before mentioned variational equations
- Γ_H is interpreted as entropy - canonical charge on the horizon
- Regularity condition of the generator implies:

$$\delta G + \delta\Gamma = R \implies \delta G = -\delta\Gamma + R$$

- So the generator is regular iff $\delta\Gamma = \delta\Gamma_\infty - \delta\Gamma_H = 0$ - first law of black hole mechanics

Calculation of entropy in Hamiltonian approach to PG

- We therefore have the algorithm to find an entropy of a black hole solution in PG:
  - We impose asymptotic conditions on the solution at infinity
  - We calculate the variation of the surface term Γ_∞
  - We interpret $\Gamma_H = T\delta S$ with T being Hawking temperature
  - We finally find entropy using the first law of black hole mechanics
- However, certain types of black holes - extremal black holes, have the property of $T = 0$, so the former approach cannot be used for them

Extremal black holes - definition

- Extremal black hole - black hole for which the parameters(mass, angular momentum and charge) are related in such a way that the inner and outer horizons coincide. It's a black hole of minimal mass such that it is compatible with its angular momentum and charge - we don't have naked singularity.
- Killing horizon - a null hypersurface on which a Killing vector ξ has norm zero - it's normal to it
- On the Killing horizon, $\partial_\mu \xi^2 = -2\kappa\xi_\mu$ holds - κ is a scalar called surface gravity
- Another definition of an extremal black hole: $\kappa = 0$ on the horizon

The notion of a near horizon geometry

- In the neighborhood of the horizon of an extremal black hole, one can introduce null coordinates (v, r, x^a), and the metric will look like:

$$ds^2 = r^2 F(r,x)dv^2 + 2dvdr + 2rh_a(r,x)dvdx^a + \gamma_{ab}(x)dx^a dx^b$$

- Here $\frac{\partial}{\partial v}$ is the Killing vector, horizon is at $r = 0$ F, h_a and γ_{ab} are continuous functions of r.
- Introduce the near-horizon limit:

$$v \rightarrow v/\epsilon \qquad r \rightarrow \epsilon r$$

with $\epsilon \longrightarrow 0$

- Near-horizon metric:

$$ds^2 = r^2 F(0,x)dv^2 + 2dvdr + 2rh_a(0,x)dvdx^a + \gamma_{ab}(x)dx^a dx^b$$

Near horizon extremal Kerr black hole

- Kerr metric:

$$ds^2 = N^2(dt+a\sin^2\theta d\phi)^2 - \frac{dr^2}{N^2} - \rho^2 d\theta^2 - \frac{\sin^2\theta}{\rho^2}\left[adt+(r^2+a^2)d\phi\right]^2$$

with:

$$N^2 = \frac{\Delta}{\rho^2} \qquad \Delta = r^2 + a^2 - 2mr \qquad \rho^2 = r^2 + a^2\cos^2\theta$$

Extremal case is obtained for $m = a$. Horizon is given by $r_+ = a$.

Near horizon extremal Kerr black hole

Near-horizon transformation:

$$\tilde{t} = \frac{\varepsilon t}{2r_+} \qquad y = \frac{\varepsilon r_+}{r - r_+} \qquad \varphi = \phi + \Omega_+ t$$

where $\varepsilon \to 0$, and $\Omega_+ = \frac{1}{2a}$ is the angular velocity of the horizon.
New metric - NHEK:

$$ds^2 = r_+^2(1+\cos^2\theta)\left[\frac{d\tilde{t}^2}{y^2} - \frac{dy^2}{y^2} - d\theta^2 - \left(\frac{2\sin\theta}{1+\cos^2\theta}\right)^2\left(d\varphi - \frac{d\tilde{t}}{y}\right)^2\right]$$

Properties of NHEK

$$ds^2 = r_+^2(1+\cos^2\theta)\left[\frac{d\tilde{t}^2}{y^2} - \frac{dy^2}{y^2} - d\theta^2 - \left(\frac{2\sin\theta}{1+\cos^2\theta}\right)^2\left(d\varphi - \frac{d\tilde{t}}{y}\right)^2\right]$$

- NHEK is not equivalent to the Kerr solution, it's a distinct solution. It is not asymptotically flat.
- It has the symmetry of an enhanced $SL(2,\mathbf{R}) \times U(1)$.
- Hypersurfaces of constant θ are warped AdS_3 spaces.

Asymptotic symmetry of NHEK

- Asymptotic boundary conditions were found by(Guica(2008)):

$$\delta g_{\mu\nu} = \begin{pmatrix} \mathcal{O}(y^{-2}) & \mathcal{O}(1) & \mathcal{O}(y) & \mathcal{O}(1) \\ & \mathcal{O}(y^{-1}) & \mathcal{O}(1) & \mathcal{O}(y^{-1}) \\ & & \mathcal{O}(y) & \mathcal{O}(y) \\ & & & \mathcal{O}(1) \end{pmatrix}$$

- Asymptotic Killing vector:

$$\xi^{\tilde{t}} = T + \mathcal{O}(y^3) \qquad \xi^{y} = y\partial_\varphi \epsilon(\varphi) + \mathcal{O}(y^2)$$
$$\xi^{\theta} = \mathcal{O}(y) \qquad \xi^{\varphi} = \epsilon(\varphi) + \mathcal{O}(y^2)$$

- Asymptotic symmetry group generated by the asymptotic Killing vectors has a conformal subgroup.

$$\xi = y\epsilon'(\varphi)\partial_y + \epsilon(\varphi)\partial_\varphi$$

Kerr solution with torsion

- It turns out that in general the Kerr solution with torsion in PG is gauge equivalent to teleparallel solution.
  It suffices to consider limiting cases of Riemannian PG and the teleparallel equivalent of GR.
- We find tetrad fields:

$$b^0 = \frac{r_+\sqrt{1+\cos\theta}}{y} d\tilde{t} \qquad b^1 = -\frac{r_+\sqrt{1+\cos\theta}}{y} dy$$

$$b^2 = r_+\sqrt{1+\cos^2\theta}\, d\theta \quad b^3 = \frac{2\sin\theta}{\sqrt{1+\cos^2\theta}} r_+ \left(d\varphi - \frac{d\tilde{t}}{y}\right)$$

Entropy of extremal Kerr black hole - boundary conditions

- Boundary conditions for the tetrad were obtained considering the metric boundary conditions derived previously

$$b^i{}_\mu = \begin{pmatrix} \mathcal{O}(y^{-1}) & \mathcal{O}(y) & \mathcal{O}(y^2) & \mathcal{O}(y) \\ \mathcal{O}(y) & \bar{b}^1{}_y + \mathcal{O}(1) & \mathcal{O}(y) & \mathcal{O}(1) \\ \mathcal{O}(y) & \mathcal{O}(1) & \bar{b}^2{}_\theta + \mathcal{O}(y) & \mathcal{O}(y) \\ \bar{b}^3{}_{\tilde{t}} f(\varphi) + \mathcal{O}(1) & \mathcal{O}(y) & \mathcal{O}(y^2) & \frac{\bar{b}^3{}_\varphi}{f(\varphi)} + \mathcal{O}(y) \end{pmatrix}$$

- Lorentz parameters that are obtained from the invariance of tetrads are all asymptotically vanishing.

Using the Hamiltonian method on NHEK

- The method of Brown and Henneaux is used. We're looking for the central charge in the algebra of improved generators:

$$\{\tilde{G}(\epsilon_1), \tilde{G}(\epsilon_2)\} = \tilde{G}(\epsilon_3) + C$$

- It can be simplified to an expression containing just the boundary corrections.

$$\{\tilde{G}(\epsilon_1), \tilde{G}(\epsilon_2)\} \approx \delta_0(\epsilon_1)\Gamma_H(\epsilon_2) \approx \Gamma_H(\epsilon_3) + C$$

- Furthermore, since C is a constant functional, it can be computed by performing variation on the background configuration.

Entropy of NHEK via Cardy formula

- Performing the variation of the boundary term at the asymptotic boundary($y = 0$) we find:

$$\delta B(\xi) = 8a_0 r_+^2 \int_0^{2\pi} (\epsilon_1 \epsilon_2' - \epsilon_2 \epsilon_1') d\varphi - 4a_0 r_+^2 \int_0^{2pi} (\epsilon_1' \epsilon_2'' - \epsilon_2' \epsilon_1'') d\varphi$$

- The second term is identified as the central charge. Expressed in Fourier modes, we have a Virasoro algebra:

$$\{L_n, L_m\} = -i(n-m)L_{m+n} - \frac{c}{12} i n^3 \delta_{n,-m}$$

- We finally obtain entropy from Cardy formula:

$$S = \sqrt{\frac{c}{6}\left(L_0 - \frac{c}{24}\right)}$$

Result for entropy of NHEK

- The entropy obtained is $S = 2\pi r_+^2$ which a smooth extremal limit of the non-extremal case.

The black holes solution that remain to be analyzed

- AdS extensions - NHEK-AdS
- Charged solutions - extremal Reissner-Nordstrom
- RN-like solutions
- Charged AdS extensions - Kerr-Newman-AdS