Quantum field theory on fuzzy de Sitter space I

Ilija Burić
University of Pisa

Quantum and Fuzzy
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based on work with Bojana Brkić, Maja Burić and Duško Latas,
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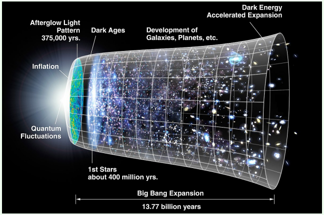
Early universe cosmology

Inflation

- Earliest stage of cosmic evolution



- Cosmological perturbations traced back to reheating surface
- Correlators very close to being scale-invariant

$$n_s = 0.9667 \pm 0.0040$$

- Inflation well approximated by weakly-coupled QFT on de Sitter
- Lends itself well to symmetry-based approaches (e.g. the bootstrap)

Cosmology and noncommutative geometry

Noncommutative cosmologies

- Quantum-gravitational effects believed to be important
- Captured by noncommutative geometry?

  1) Moyal-like deformations $\rightarrow$ inhomogeneities in the CMB

  2) matrix models $\rightarrow$ singularity resolution

  3) minisuperspace, spectral action, modified dispersion relations ...
- Today: Fuzzy de Sitter space

  $\rightarrow$ agnostic about microscopic physics

  $\rightarrow$ has $SO(1,4)$ symmetry (broken/twisted)

History of fuzzy de Sitter space

Noncommutative frames

- Constructed within the noncommutative frame formalism

$$[\hat{p}_\alpha, \hat{x}^\mu] = e^\mu_\alpha(\hat{x}), \qquad [\hat{p}_\alpha, \hat{p}_\beta] = A_{\alpha\beta}{}^{\gamma\delta}\hat{p}_\gamma\hat{p}_\delta + C_{\alpha\beta}{}^{\gamma}\hat{p}_\gamma + B_{\alpha\beta}$$

- Rigid framework ensuring ‘correct’ differential geometry
- Gives rise to classical and quantum field theory
- Spectral properties, commutative limit?

Fuzzy de Sitter model

- Introduced in [Burić, Madore] - coordinates and momenta
- Spectra of coordinates [Burić, Latas, Nenadović]
- Riemannian geometry and quantum mechanics [Brkić, Burić, Latas]

The role of fuzzy harmonics

Fuzzy spherical harmonics

- Building blocks of fuzzy field theories

$$Y_{0,0} = \frac{\mathrm{I}}{2\sqrt{\pi}}, \qquad Y_{1,0} = \frac{\sigma_3}{2\sqrt{\pi}}, \qquad Y_{1,\pm 1} = \frac{\sigma_1 \pm i\sigma_2}{2\sqrt{2\pi}}$$

- Close under multiplication

$$Y_{l_1,m_1} Y_{l_2,m_2} = \sum_{l,m}^{s} C^{l_1 l_2 l}_{m_1 m_2 m} Y_{l,m}$$

- Serve for field expansion

$$\Phi = \sum_{l,m} \frac{1}{\sqrt{2\omega_l}} \left(a_{l,m} e^{-i\omega_l t} Y_{l,m} + a^{\dagger}_{l,m} e^{i\omega_l t} Y^{*}_{l,m} \right)$$

Aim for this talk

Harmonics continued

- In interacting theories

$$S = \frac{4\pi}{N} \int dt \, \mathrm{tr} \left(\frac{1}{2}\dot{\Phi}^2 + \Phi[p^a, [p_a, \Phi]] - \frac{\lambda}{4!}\Phi^4 \right)$$

structure constants determine the spectrum

$$H = \sum_{l,m} \omega_l a^\dagger_{l,m} a_{l,m}$$

$$+ \frac{\pi\lambda}{24N} \sum \frac{C^{l_1 l_2 l}_{m_1 m_2 m} C^{l_4 l_3 l}_{m_4 m_3 m}}{\sqrt{\omega_{l_1} \dots \omega_{l_4}}} (a_{l_1,m_1} + (-1)^{m_1} a^\dagger_{l_1,-m_1}) \dots ((-1)^{m_4} a_{l_4,-m_4} + a^\dagger_{l_4,m_4})$$

Today

- Quantisation of commutative field modes
- Compatibility with the Bunch-Davies vacuum
- Complete set of fuzzy harmonics

Plan for the talk

1. Introduction
2. Quantum field theory on de Sitter space
3. Fuzzy de Sitter harmonics by separation of variables
4. Higher Kaluza-Klein modes
5. Conclusion

De Sitter space

Symmetries and boundary

- Poincaré coordinates

$$ds^2 = \frac{1}{\eta^2}\left(-d\eta^2 + dx^i dx_i\right), \qquad \eta < 0$$

- Asymptotic boundary = reheating surface : $\eta \to 0$
- Isometry group $G = SO(1,4)$

$$P_i = -\partial_{x^i}, \quad L_{ij} = x_i\partial_{x^j} - x_j\partial_{x^i},$$
$$D = -\eta\partial_\eta - x^i\partial_{x^i}, \quad K_i = (\eta^2 - x^2)\partial_{x^i} - 2x_i D$$

  acts as the conformal group on the boundary
- Alternative 'physical' (as opposed to 'comoving') coordinates

$$y^i = \frac{x^i}{\eta}$$

  such that the dilation generator is $D = -\eta\partial_\eta$

Quantisation

Field modes and vacua

- Klein-Gordon equation

$$\left(\Delta_{dS_4} - M^2\right)\Phi = 0$$

- Quantisation: choice of positive frequency modes

$$\Phi = \sum_i \left(a_i v_i + a_i^\dagger v_i^*\right), \qquad a_i|0\rangle = 0$$

subject to

$$\langle v_i, v_j\rangle = \delta_{ij}, \quad \langle v_i^*, v_j^*\rangle = -\delta_{ij}, \quad \langle v_i, v_j^*\rangle = 0$$

- Solutions carry a representation of $SO(1,4)$

$$\Phi(\eta, x^i) \sim (-\eta)^{\Delta_+}\Phi^+(x^i) + (-\eta)^{\Delta_-}\Phi^-(x^i), \qquad \eta \to 0$$

consisting of two irreducible components

$$\Delta_\pm = \frac{3}{2} \pm i\kappa, \qquad \kappa^2 = M^2 - \frac{9}{4}$$

Separation of variables

Field modes and vacua

- Special vacua:

  1) $SO(1,4)$-invariant $\rightarrow$ complex parameter λ [Chernikov, Tagirov; Allen]

  2) Bunch-Davies $\rightarrow$ flat space behaviour of the two-point function

- Separation of variables in (η, y^i) compatible with $SO(1,4)$

$$v(\eta, y^i) = (-\eta)^{-i\omega} F(y^2)\, \Psi_l^m(\theta,\phi)\,,$$

- Field modes on the boundary

$$v_{\omega,l,m,\kappa} \cong \begin{pmatrix} c\, r^{-\frac{3}{2}-i\kappa-i\omega}\, \Psi_l^m(\theta,\phi) \\ 0 \end{pmatrix}, \quad v_{\omega,l,m,-\kappa} \cong \begin{pmatrix} 0 \\ c\, r^{-\frac{3}{2}+i\kappa-i\omega}\, \Psi_l^m(\theta,\phi) \end{pmatrix}$$

- α-vacuum in the sense of [Allen] with

$$\coth\alpha = e^{\pi\kappa}$$

- Bogoliubov transformation to Poincaré modes understood

Fuzzy de Sitter space

Basic properties

- De Sitter as the hyperboloid

$$-X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 = \ell^2$$

- Reminiscent of quartic Casimir of $SO(1,4)$

$$-W_0^2 + W_1^2 + W_2^2 + W_3^2 + W_4^2 = C_4$$

- Coordinates and momenta

$$\hat{\eta} = -\ell(W_0 - W_4), \quad \hat{x}^i = \ell W^i \qquad \hat{p}_0 = D, \quad \hat{p}_i = -P_i$$

- Frame relations and momentum commutators

$$[\hat{p}_\mu, \hat{x}^\nu] = \delta_\mu^\nu \, \hat{\eta} \qquad [\hat{p}_0, \hat{p}_i] = \hat{p}_i, \quad [\hat{p}_i, \hat{p}_j] = 0$$

- Covariant differential geometry $\rightarrow$ Einstein manifold
- Laplace-Beltrami operator

$$\Delta\Phi = -[\hat{p}_0, [\hat{p}_0, \Phi]] + 3[\hat{p}_0, \Phi] + [\hat{p}_i, [\hat{p}_i, \Phi]]$$

Two-dimensional case

In various dimensions

- $d = 6, 8, \dots$: via Pauli-Lubanski vector [Burić, Madore]
- $d = 3$: the fuzzy BTZ black hole [IB, Burić]

Deformed dS_2

- $d = 2$: h-deformed hyperbolic plane [Cho; Madore, Steinacker]

$$\hat{\eta} = -i\hbar P, \quad \hat{x} = -i\hbar D, \qquad \hat{p}_0 = D, \quad \hat{p}_1 = -P$$

- Closely analogous differential calculus

$$[\hat{p}_0, \hat{\eta}] = \hat{\eta}, \quad [\hat{p}_1, \hat{x}] = \hat{\eta}, \qquad [\hat{p}_0, \hat{p}_1] = \hat{p}_1$$

leading to the Laplacian

$$\Delta\Phi = -[\hat{p}_0, [\hat{p}_0, \Phi]] + [\hat{p}_0, \Phi] + [\hat{p}_1, [\hat{p}_1, \Phi]]$$

Harmonics in separated variables

Commutators in $(\hat{\eta}, \hat{y})$ variables

- Variable y defined by symmetrisation

$$\hat{y} = \frac{1}{2}\left(\hat{\eta}^{-1}\hat{x} + \hat{x}\,\hat{\eta}^{-1}\right), \qquad [\hat{\eta}, \hat{y}] = i\hbar$$

- Momenta preserve ordering

$$[\hat{p}_0, \hat{\eta}] = \hat{\eta}\,, \qquad [\hat{p}_1, \hat{\eta}] = 0\,,$$
$$[\hat{p}_0, \hat{y}] = -\hat{y}\,, \quad [\hat{p}_1, \hat{y}] = 1\,.$$

- Look for harmonics of the form

$$\Phi(\hat{\eta}, \hat{y}) = (-\hat{\eta})^{-i\omega} f(\hat{y})$$

- Solutions by putting hats to classical ones

$$\hat{v}_{\omega,\kappa}(\hat{\eta}, \hat{y}) = c_{\omega,\kappa}\,(-\hat{\eta})^{-i\omega}\left(\hat{y}^2 - 1\right)^{-\frac{i\omega}{2}} Q^{-i\omega}_{-\frac{1}{2}+i\kappa}(\hat{y})$$

Solutions in four dimensions

Solutions in four dimensions

- Coordinates

$$\hat{y}^i = \frac{1}{2}\left(\hat{\eta}^{-1}\hat{x}^i + \hat{x}^i\hat{\eta}^{-1}\right)$$

- Commutation relations

$$[\hat{p}_0, \hat{\eta}] = \hat{\eta}\,, \qquad [\hat{p}_i, \hat{\eta}] = 0\,,$$
$$[\hat{p}_0, \hat{y}^j] = -\hat{y}^j\,, \quad [\hat{p}_i, \hat{y}^j] = \delta_i^j$$

- Separated solutions

$$v = (-\eta)^{-i\omega} \sum c_{ijk}\, y_1^i y_2^j y_3^k \quad \mapsto \quad \hat{v} = (-\hat{\eta})^{-i\omega} \sum c_{ijk}\, \hat{y}_1^i \hat{y}_2^j \hat{y}_3^k$$

Questions

- A different ordering of $\hat{y}_i$?
- Completeness of solutions?

Fuzzy functions as integral kernels

Functions as differential operators

- Principal series representations

$$P_i = -\partial_i, \quad L_{ij} = z_i\partial_j - z_j\partial_i - \Sigma_{ij}\,,$$
$$D = -z^i\partial_i - \Delta, \quad K_i = -z^2\partial_i - 2z_i D - 2z^j\Sigma_{ij}$$

acting on spinor fields $\psi_a(z^i)$

- Fuzzy functions elements of the tensor product

$$\mathcal{A} = \mathrm{End}(\mathcal{H}) \cong \mathcal{H} \otimes \mathcal{H}^*\,,$$

- Action of momenta via commutators becomes

$$\mathrm{ad}_X \mapsto X^L + X^R\,.$$

and gives rise to the fuzzy Laplacian

$$\Delta = -\left(D^L + D^R\right)^2 + (d-1)\left(D^L + D^R\right) + \sum_{i=1}^{d-1}\left(P_i^L + P_i^R\right)^2$$

Working in time eigenbasis

Fuzzy harmonics as matrices

- Coordinates as differential operators

$$\hat{\eta} = i\bar{k}\partial_z \,, \quad \hat{x} = i\bar{k}(z\partial_z + \Delta)$$

- Eigenbasis of $\hat{\eta}$ = Fourier space

$$P = -iq, \quad D = q\partial_q + 1 - \Delta, \quad K = iq\partial_q^2 - 2iD\partial_q$$

- Eigenstates of the radial coordinates

$$\hat{y}|\lambda\rangle = \lambda|\lambda\rangle, \qquad \langle q|\lambda\rangle = \frac{1}{\sqrt{2\pi}}\, q^{-\frac{1}{2}+\Delta}\, e^{-i\lambda q}$$

- Harmonics as functions of two variables

$$\langle \eta_L|\hat{v}_{\omega,\kappa}|\eta_R\rangle = \frac{C_{\omega,\kappa}}{\bar{k}^{i\omega+1/2}}\, \eta_L^{i\tau-i\omega}\eta_R^{-i\tau}\,(\eta_L-\eta_R)^{i\omega-\frac{1}{2}}\, J_{i\kappa}\left(\frac{\eta_L-\eta_R}{\bar{k}}\right)$$

- Fully characterised by a pair of differential equations
- Similar structure in four dimensions

General solution to Klein-Gordon equation

The fuzzy Laplacian is integrable

- Work in coordinates

$$z_L^i = y^i + \xi^i, \quad z_R^i = y^i - \xi^i$$

in which the momenta assume the form

$$P_i^L + P_i^R = -\partial_{y^i}, \quad D^L + D^R = -y^i \partial_{y^i} - \xi^i \partial_{\xi^i} - 3$$

- Compute the Laplacian and observe one has solutions

$$g(\vartheta)(-\xi)^\delta f(y^i)\, e_a \otimes e_b$$

- Reduced operator is closely related to commutative Laplacian

$$w^{l',m',ab}_{\omega,l,m,\kappa}(z_L^i, z_R^j) = Y_{l'}^{m'}(\vartheta)\,(-\xi)^{i\omega-3}\,\rho^{-\frac{3}{2}-i(\omega+\kappa)}$$

$$\times\ {}_2F_1\left(\frac{2l+3+2i(\omega+\kappa)}{4}, \frac{-2l+1+2i(\omega+\kappa)}{4}; 1+i\kappa; \rho^{-2}\right) Y_l^m(\theta)\, e^a \otimes e^b$$

- Commutative solutions with a tower of modes labelled by (l', m')

Summary and perspectives

Summary of results

- Complete set of harmonics on fuzzy de Sitter spaces in $d = 2, 4$
- Separation of variables, vacuum invariance, higher-spin modes

Future directions

- Details of the Kaluza-Klein structure
  $\rightarrow$ spinning harmonics via weight-shifting operators
  $\rightarrow$ suppression of higher modes
- Quantum field theory
  $\rightarrow$ propagator
  $\rightarrow$ coherent-like states
- Extensions
  $\rightarrow$ fuzzy BTZ the simplest model to study
  $\rightarrow$ other spaces in four dimensions? AdS-Kerr black hole?

Thank you for your attention.

Srećan rodjendan, Majče!

Propagator

Field expansion

$$\Phi = \int d\omega \left(a_\omega \hat{v}_\omega + a_\omega^\dagger \hat{v}_\omega^* \right)$$

Two-point function

$$\langle \Phi_1 \Phi_2 \rangle = \int d\omega \; \hat{v}_\omega \hat{v}_\omega^*$$

leads to a solvable integral. Another option

$$\hat{G}_2 = \hat{G}_2(\hat{\eta}, \hat{x}, \hat{\eta}', \hat{x}') = \frac{\Gamma\left(\frac{1}{2} + i\kappa\right)\Gamma\left(\frac{1}{2} - i\kappa\right)}{4\pi} \, {}_2F_1\left(\frac{1}{2} + i\kappa, \frac{1}{2} - i\kappa; 1; \frac{1 + \hat{Z}}{2}\right)$$

Quantum geodesic distance

$$\hat{Z} = \frac{1}{2}\left(\hat{\eta}\hat{\eta}'^{-1} + \hat{\eta}^{-1}\hat{\eta}' + 2\hat{y}\hat{y}' - \hat{\eta}'^{-1}\{\hat{\eta}^{-1}, \hat{x}^2\} - \hat{\eta}^{-1}\{\hat{\eta}'^{-1}, \hat{x}'^2\}\right)$$

$$= \frac{q}{q'}\partial_q^2 + \frac{q'}{q}\partial_{q'}^2 + \left(\frac{3}{2} - \Delta\right)\left(\frac{1}{q'}\partial_q + \frac{1}{q}\partial_{q'}\right) + \frac{\frac{1}{2}(q^2 + q'^2) + \Delta(\Delta - 1) + \frac{3}{4}}{qq'}$$

Semi-classical states

Start with vector $|\xi_0\rangle$ with expectation values

$$\eta_0 = \langle \xi_0 | \hat{\eta} | \xi_0 \rangle, \qquad x_0 = \langle \xi_0 | \hat{x} | \xi_0 \rangle$$

$\xi_0 = (\eta_0, x_0)$ a classical point. Another point obtained by

$$\xi_1 = \lambda^{-e_0} e^{-be_1} \xi_0 = \left(\lambda^{-1} \eta_0, x_0 - b\eta_0 \right)$$

$\{e_0, e_1\}$ is the classical frame. Semi-classical state 'associated with ξ_1' by

$$|\xi_1\rangle = \lambda^{\hat{p}_0} e^{b\hat{p}_1} |\xi_0\rangle \ .$$

Local measurements at ξ_1 modelled by expectation values of operators in $|\xi_1\rangle$. Axioms of the frame formalism ensure consistency

$$\langle \xi_1 | \hat{\eta} | \xi_1 \rangle = \eta_1, \qquad \langle \xi_1 | \hat{x} | \xi_1 \rangle = x_1$$

Symmetry breaking

Classical frame

$$e_0 = \eta\partial_\eta, \quad e_i = \eta\partial_{x^i}$$

Carried to $SO(1,d)$ generators by

$$\Phi : \mathrm{dS}_d \to \mathrm{dS}_d, \qquad (\eta, x^i) \mapsto \left(\eta^{-1}, \frac{x^i}{\eta}\right)$$

$$\Phi^* e_0 = P_0, \quad \Phi^* e_i = P_i \ .$$

Manifested on the noncommutative level in differential equations satisfied by the modes

$$\hat{D}\hat{v}_{\omega,\kappa} = -i\omega\hat{v}_{\omega,\kappa}, \qquad \hat{D} = \left(D^L - \frac{1}{2}\right) - \frac{\eta_L}{\eta_R}\left(D^R - \frac{1}{2}\right) \ .$$