Entropy of black holes coupled to a scalar field

Branislav Cvetković[1]

[1]Institute of physics, Belgrade

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Talk overview

The talk is based on the following papers

- M. Blagojević and B. Cvetković, Entropy in Poincaré gauge theory: Hamiltonian approach, Phys. Rev. D **99**, 104058 (2019)
- M. Blagojević and B.Cvetković, Entropy of Kerr-Newman-AdS black holes with torsion, Phys.Rev.D **105** (2022) 104014
- M. Blagojević and B.Cvetković, Entropy of black holes coupled to a scalar field, Phys. Rev. D **107** (2023) 10, 104022
- M. Blagojević and B.Cvetković, Thermodynamics of charged black hole with scalar hair, Phys.Rev.D **108** (2023) 12, 124012

- Already in 1960s, Kibble and Sciama proposed a new theory of gravity, the Poincaré gauge theory (PG), based on gauging the Poincaré group of spacetime symmetries.
- PG is characterized by a Riemann-Cartan (RC) *geometry* of spacetime, in which both the torsion and the curvature are essential ingredients of the *gravitational dynamics*.
- Nowadays, PG is a well-established approach to gravity, representing a natural gauge-field-theoretic extension of general relativity (GR).
- In the past half century, many investigations of PG have been aimed at clarifying different aspects of both the geometric and dynamical roles of torsion. In particular, successes in constructing exact solutions with torsion naturally raised the question of how their *conserved charges* are influenced by the presence of torsion.

- We shall reconsider the notion of conserved charge in the Hamiltonian formalism, as it represents the most natural basis for the main subject of the present talk, the *influence of torsion* on black hole entropy.
- The expressions for the conserved charges in PG were first found for asymptotically flat solutions. In the Hamiltonian approach to PG the conserved charges are represented by a boundary term, defined by requiring the variation of the canonical gauge generator to be a well-defined (differentiable) functional on the phase space.
- A covariant version of the Hamiltonian approach, introduced later by Nester, turned out to be an important step in understanding the conservation laws. This was clearly demonstrated by Hecht and Nester, in their analysis of the conserved charges for asymptotically flat or (A)dS.

- Despite an intensive activity in exploring the notion of conserved charges in the *generic* four-dimensional (4D) PG, systematic studies of black hole entropy in the presence of torsion have been largely neglected so far.
- One should mention here an early and general proposal by Nester which did not prove to be quite successful.
- Later investigations were restricted to EC theory, which is certainly not sufficient to justify any conclusion on the general relation between torsion and entropy.
- In 3D gravity, black hole entropy is well understood for solutions possessing the asymptotic conformal symmetry.
- The physics of black holes is an arena where thermodynamics, gravity, and quantum theory are connected through the existence of entropy as an intrinsic dynamical aspect of black holes.

- In the 1990s, understanding of the *classical* black hole entropy reached a level that can be best characterized by Wald's words: “Black hole entropy is the Noether charge" .
- The question that we wish to address is whether such a challenging idea can improve our understanding of black hole entropy in PG.
- We constructed the canonical gauge generator in the first order formulation of PG, which improved form is used to obtain the variational equation for the asymptotic canonical charge, located at the spatial 2-boundary at infinity.
- Following the idea that “entropy is the canonical charge at horizon," we are led to define black hole entropy by the same variational equation, located at black hole horizon.
- The differentiability of the gauge generator guarantees the validity of the first law of black hole thermodynamics.

Notations and conventions

- Our conventions are as follows.
- The greek indices $(\mu, \nu, \dots)$ refer to the coordinate frame, with a time-space splitting expressed by $\mu = (0, \alpha)$.
- The latin indices $(i, j, \dots)$ refer to the local Lorentz frame.
- ϑ^i is the orthonormal tetrad (1-form), e_i is the dual basis (frame), with $e_i \rfloor \vartheta^k = \delta_i^k$, and the Lorentz metric is $\eta_{ij} = (1, -1, -1, -1)$.
- The volume 4-form is $\hat{\epsilon} = \vartheta^0\vartheta^1\vartheta^2\vartheta^3$, the Hodge dual of a form α is ${}^\star\alpha$, with ${}^\star 1 = \hat{\epsilon}$, and the totally antisymmetric tensor is defined by ${}^\star(\vartheta_i\vartheta_j\vartheta_m\vartheta_n) = \varepsilon_{ijmn}$, where $\varepsilon_{0123} = +1$.
- The exterior product of forms is implicit.

- Basic dynamical variables of PG are the tetrad field ϑ^i and the spin connection ω^{ij} (1-forms), the gauge potentials related to the translation and the Lorentz subgroups of the Poincaré group, respectively. The corresponding field strengths are the torsion $T^i = d\vartheta^i + \omega^i{}_m \vartheta^m$ and the curvature $R^{ij} = d\omega^{ij} + \omega^i{}_m \omega^{mj}$ (2-forms).
- Varying the gravitational Lagrangian $L_G = L_G(\vartheta^i, T^i, R^{ij})$ (4-form) with respect to ϑ^i and ω^{ij} yields the gravitational field equations *in vacuum*. After introducing the covariant field momenta, $H_i := \partial L_G / \partial T^i$ and $H_{ij} := \partial L_G / \partial R^{ij}$, and the associated energy-momentum and spin currents, $E_i := \partial L_G / \partial \vartheta^i$ and $E_{ij} := \partial L_G / \partial \omega^{ij}$, the equations read

$$\delta\vartheta^i : \qquad \nabla H_i + E_i = 0\,, \tag{2.1a}$$

$$\delta\omega^{ij} : \qquad \nabla H_{ij} + E_{ij} = 0\,. \tag{2.1b}$$

- Assuming the gravitational Lagrangian L_G to be at most quadratic in the field strengths and parity invariant,

$$L_G = -{}^\star(a_0 R + 2\Lambda) + T^i \sum_{n=1}^{3} {}^\star(a_n\, {}^{(n)}T_i) + \frac{1}{2} R^{ij} \sum_{n=1}^{6} {}^\star(b_n\, {}^{(n)}R_{ij}) \,,$$

the gravitational field momenta take the form

$$H_i = 2 \sum_{m=1}^{3} {}^\star(a_m\, {}^{(m)}T_i) \,, \quad H_{ij} = -2a_0 {}^\star(\vartheta^i \vartheta^j) + H'_{ij} \,,$$

$$H'_{ij} := 2 \sum_{n=1}^{6} {}^\star(b_n\, {}^{(n)}R_{ij}) \,.$$

- Here, (a_0, a_m, b_n) are the Lagrangian parameters, with $16\pi a_0 = 1$, Λ is a cosmological constant, and ${}^{(m)}T_i$ and ${}^{(n)}R_{ij}$ are irreducible parts of torsion and curvature.

- A black hole can be described as a region of spacetime which is causally disconnected from the rest of spacetime.
- The boundary of a black hole is a null hypersurface, known as the *event horizon*.
- Let us consider a black hole characterized by the existence of a Killing vector field ξ. A null hypersurface to which the Killing vector is normal, is called the *Killing horizon* ($\mathcal{K}$). As a consequence, $\xi^2 := g_{\mu\nu}\xi^\mu\xi^\nu = 0$ on $\mathcal{K}$. The gradient $\partial_\mu(\xi^2)$ is also normal to $\mathcal{K}$ and it must be proportional to ξ_μ,

$$\partial_\mu(\xi^2) = -2\kappa\xi_\mu\,, \tag{2.2}$$

  where the scalar function κ is known as *surface gravity*.
- One can show, without making use of any field equations, that for a wide class of stationary black holes (systems in “equilibrium"), the Killing horizon coincides with event horizon.

- The essential property of surface gravity is expressed by the *zeroth law* of black hole mechanics: For a wide class of stationary black holes, surface gravity is constant over the entire event horizon.
- Since null geodesics and Killing vector fields are purely metric notions, they can be directly transferred to PG. Thus, the form of surface gravity and the associated zeroth law of black mechanics are also valid in PG.
- The calculation of κ should be done in coordinates that are well defined on the outer horizon, such as ingoing Edington-Finkelstein coordinates, where the metric reads

$$ds^2 = N^2 dv^2 - 2dv\,dr - r^2 d\Omega^2\,, \qquad N = N(r)\,, \tag{2.3}$$

while the definition (2.2) of surface gravity takes the form

$$\partial_r N^2 = 2\kappa\,. \tag{2.4}$$

- In PG, the conserved charges are determined as the values of the (improved) canonical generators of spacetime symmetries, associated to suitable asymptotic conditions.
- The canonical procedure is simplified by transforming the quadratic Lagrangian into the “first order" form

$$L_G = T^i\tau_i + \frac{1}{2}R^{ij}\rho_{ij} - V(\vartheta^i, \tau_i, \rho_{ij})\,, \qquad (3.1)$$

where the gravitational potentials $(\vartheta^i, \omega^{ij})$ and “covariant momenta" (τ_i, ρ_{ij}), are *independent* dynamical variables.

- The potential V is a quadratic function of (τ_i, ρ_{ij}) which ensures the on-shell relations $\tau_i = H_i$ and $\rho_{ij} = H_{ij}$.
- In the tensor formalism, the Lagrangian density reads

$$\tilde{\mathcal{L}}_G = -\frac{1}{4}\varepsilon^{\mu\nu\lambda\rho}\left(T^i{}_{\mu\nu}\tau_{i\lambda\rho} + \frac{1}{2}R^{ij}{}_{\mu\nu}\rho_{ij\lambda\rho}\right) - \tilde{\mathcal{V}}(\vartheta, \tau, \rho)\,. \qquad (3.2)$$

- The gravitational field equations (in vacuum) are obtained by varying $\tilde{\mathcal{L}}_G$ with respect to the independent dynamical variables $\vartheta^i{}_\mu, \omega^{ij}{}_\mu, \tau^i{}_{\mu\nu}$ and $\rho^{ij}{}_{\mu\nu}$:

$$\nabla_\mu {}^{(\star)}\tau_i{}^{\mu\nu} - \frac{\partial\tilde{\mathcal{V}}}{\partial b^i{}_\nu} = 0\,, \tag{3.3a}$$

$$2\vartheta_{[j\mu}{}^{(\star)}\tau_{i]}{}^{\mu\nu} + \nabla_\mu \rho_{ij}{}^{\mu\nu} = 0\,, \tag{3.3b}$$

$$-{}^{(\star)}T^{i\mu\nu} - \frac{\partial\tilde{\mathcal{V}}}{\partial \tau_{i\mu\nu}} = 0\,, \tag{3.3c}$$

$$-{}^{(\star)}R^{ij\mu\nu} - \frac{\partial\tilde{\mathcal{V}}}{\partial \rho_{ij\mu\nu}} = 0\,, \tag{3.3d}$$

where we use the notation ${}^{(\star)}\tau_i{}^{\mu\nu} := \frac{1}{2}\varepsilon^{\mu\nu\lambda\rho}\tau_{i\lambda\rho}$, and similarly for ${}^{(\star)}\rho_{ij}{}^{\mu\nu}$, ${}^{(\star)}T^{i\mu\nu}$ and ${}^{(\star)}R^{ij\mu\nu}$.

- Starting with the field variables $\varphi^A = (\vartheta^i{}_\mu, \omega^{ij}{}_\mu, \tau^i{}_{\mu\nu}, \rho^{ij}{}_{\mu\nu})$ and the corresponding canonical momenta $\pi_A = (\pi_i{}^\mu, \pi_{ij}{}^\mu, P_i{}^{\mu\nu}, P_{ij}{}^{\mu\nu})$, one obtains the following primary constraints:

$$
\begin{aligned}
&\phi_i{}^0 := \pi_i{}^0 \approx 0\,, && \phi_i{}^\alpha := \pi_i{}^\alpha + {}^{(\star)}\tau_i{}^{0\alpha} \approx 0\,,\\
&\phi_{ij}{}^0 := \pi_{ij}{}^0 \approx 0\,, && \phi_{ij}{}^\alpha := \pi_{ij}{}^\alpha + \frac{1}{2}\,{}^{(\star)}\rho_{ij}{}^{0\alpha} \approx 0\,,\\
&P_i{}^{\mu\nu} \approx 0\,, && P_{ij}{}^{\mu\nu} \approx 0\,.
\end{aligned} \tag{3.4}
$$

The canonical Hamiltonian is found to have the form

$$
\begin{aligned}
&H_c := \vartheta^i{}_0\mathcal{H}_i + \frac{1}{2}\omega^{ij}{}_0\mathcal{H}_{ij} + \tau_{i0\alpha}{}^{(\star)}T^{i0\alpha} + \frac{1}{2}\rho_{ij0\alpha}{}^{(\star)}R^{ij0\alpha} + \tilde{\mathcal{V}}\,,\\
&\mathcal{H}_i := \nabla_\alpha{}^{(\star)}\tau_i{}^{0\alpha}\,,\\
&\mathcal{H}_{ij} := 2\vartheta_{[j\alpha}{}^{(\star)}\tau_{i]}{}^{0\alpha} + \nabla_\alpha{}^{(\star)}\rho_{ij}{}^{0\alpha}\,.
\end{aligned} \tag{3.5}
$$

- The total Hamiltonian reads

$$H_T := H_c + u^i{}_\mu \phi_i{}^\mu + \frac{1}{2} u^{ij}{}_\mu \phi_{ij}{}^\mu + \frac{1}{2} v^i{}_{\mu\nu} P_i{}^{\mu\nu} + \frac{1}{4} v^{ij}{}_{\mu\nu} P_{ij}{}^{\mu\nu} \,,$$

where u's and v's are canonical multipliers.

- The consistency conditions of the sure primary constraints produces the secondary constraints

$$\hat{\mathcal{H}}_i := \mathcal{H}_i + \frac{\partial \tilde{\mathcal{V}}}{\partial \vartheta^i{}_0} \approx 0 \,,\; \hat{\mathcal{H}}_{ij} := \mathcal{H}_{ij} \approx 0 \,,$$

$$\hat{\mathcal{T}}^{i0\alpha} := {}^{(\star)}T^{i0\alpha} + \frac{\partial \tilde{\mathcal{V}}}{\partial \tau_{i0\alpha}} \approx 0 \,,\; \hat{\mathcal{R}}^{ij0\alpha} := {}^{(\star)}R^{ij0\alpha} + \frac{\partial \tilde{\mathcal{V}}}{\partial \rho_{ij0\alpha}} \approx 0 \,,$$

which correspond to certain components of the field equations (3.3).

- The remaining primary constraints are second class.

- We can construct the corresponding DB and use them in the consistency procedure on the reduced phase space:

$$\{\vartheta^i{}_\alpha, \tau_{j\beta\gamma}\}^* = \delta^i_j \varepsilon_{0\alpha\beta\gamma}\,, \qquad \{\omega^{ij}{}_\alpha, \rho_{kl\beta\gamma}\}^* = \delta^{[i}_k \delta^{j]}_l \varepsilon_{0\alpha\beta\gamma}\,.$$

- The form of the total Hamiltonian is simplified:

$$H_T = H_c + u^i{}_0 \pi_i{}^0 + \frac{1}{2} u^{ij}{}_0 \pi_{ij}{}^0 + v^i{}_{0\beta} P_i{}^{0\beta} + \frac{1}{2} v^{ij}{}_{0\beta} P_{ij}{}^{0\beta}\,. \qquad (3.6)$$

- In terms of the secondary constraints H_c reads

$$H_c = \vartheta^i{}_0 \hat{\mathcal{H}}_i + \frac{1}{2} \omega^{ij}{}_0 \mathcal{H}_{ij} + \tau_{i0\alpha} \hat{\mathcal{T}}^{i0\alpha} + \frac{1}{2} \rho_{ij0\alpha} \hat{\mathcal{R}}^{ij0\alpha}. \qquad (3.7)$$

- A phase-space functional G is a good gauge generator if it generates the correct gauge transformations of all phase-space variables.

- Relying on an explicit construction of G in 3D PG, we display here its generalization to 4D:

$$G[\xi,\theta]=\int_\Sigma d^3x\left(G_1+G_2\right),\, G_2=\frac{1}{2}\dot{\theta}^{ij}\pi_{ij}{}^0+\frac{1}{2}\theta^{ij}\mathcal{M}_{ij}\,,$$

$$G_1=\dot{\xi}^\mu\Big(\vartheta^i{}_\mu\pi_i{}^0+\frac{1}{2}\omega^{ij}{}_\mu\pi_{ij}{}^0+\tau^i{}_{\mu\beta}P_i{}^{0\beta}+\frac{1}{2}\rho^{ij}{}_{\mu\beta}P_{ij}{}^{0\beta}\Big)+\xi^\mu\mathcal{P}_\mu\,,$$

$$\mathcal{P}_\mu:=\vartheta^i{}_\mu\hat{\mathcal{H}}_i+\frac{1}{2}\omega^{ij}{}_\mu\mathcal{H}_{ij}+\tau^i{}_{\mu\beta}\hat{\mathcal{T}}_i{}^{0\beta}+\frac{1}{2}\rho^{ij}{}_{\mu\beta}\hat{\mathcal{R}}_{ij}{}^{0\beta}$$

$$+(\partial_\mu\vartheta^i{}_0)\pi_i{}^0+\frac{1}{2}(\partial_\mu\omega^{ij}{}_0)\pi_{ij}{}^0+(\partial_\mu\tau^i{}_{0\beta})P_i{}^{0\beta}+\frac{1}{2}(\partial_\mu\rho^{ij}{}_{0\beta})P_{ij}{}^{0\beta}$$

$$-\partial_\beta\left(\tau^i{}_{0\mu}P_i{}^{0\beta}+\frac{1}{2}\rho^{ij}{}_{0\mu}P_{ij}{}^{0\beta}\right)\,,$$

$$\mathcal{M}_{ij}:=\mathcal{H}_{ij}+2\left(\vartheta_{[i0}\pi_{j]}{}^0+\omega^k{}_{[i0}\pi_{kj]}{}^0+\tau_{[i0\gamma}P_{j]}{}^{0\gamma}+\rho^k{}_{[i0\gamma}P_{kj]}{}^{0\gamma}\right)\,.$$

- The Hamiltonian formulation of gravity is based on the existence of a family of spacelike hypersurfaces Σ, labeled by the time parameter t. Each Σ is bounded by a closed 2-surface at spatial infinity, which is used to define the *asymptotic charge*. When Σ is a black hole manifold, it also possesses an “interior" boundary, the horizon, which serves to define *black hole entropy*.
- In PG, conserved charges are closely related to the canonical gauge generator G. Since G acts on dynamical variables via the PB (or DB) operation, it should have well-defined functional derivatives. If G does not satisfy this requirement the problem can be solved by adding a suitable surface term Γ_∞, located at the boundary of Σ at infinity, such that $\tilde{G} = G + \Gamma_\infty$ is well defined. The value of Γ_∞ is exactly the canonical charge of the system.

- Any particular solution of PG is characterized by a set of asymptotic conditions. Demanding that local Poincaré transformations preserve these conditions, one obtains certain restrictions on the Killing-Lorentz parameters. The restricted parameters define the asymptotic symmetry, which is essential for the existence of conserved charges.
- We consider the variation of the gauge generator

$$
\begin{aligned}
\delta G &= \int_\Sigma d^3x(\delta G_1 + \delta G_2)\,, \\
\delta G_1 &= \xi^\mu \Big[\vartheta^i{}_\mu \delta\hat{\mathcal{H}}_i + \frac{1}{2}\omega^{ij}{}_\mu \delta\mathcal{H}_{ij} + \tau_{i\mu\alpha}\delta\hat{\mathcal{T}}^{i0\alpha} + \frac{1}{2}\rho_{ij\mu\alpha}\delta\hat{\mathcal{R}}^{ij0\alpha}\Big]\,, \\
\delta G_2 &= \frac{1}{2}\theta^{ij}\delta\mathcal{H}_{ij} + R\,,
\end{aligned}
\tag{4.1}
$$

where δ is the variation over the set of asymptotic states, and R denotes regular (differentiable) terms.

- To get rid of the unwanted $\delta\partial_\mu\varphi$ terms which spoil the differentiability of G, one can perform a partial integration,

$$\delta G_1 = \frac{1}{2}\varepsilon^{0\alpha\beta\gamma}\partial_\alpha\Big\{\xi^\mu\Big[\vartheta^i{}_\mu\delta\tau_{i\beta\gamma} + \frac{1}{2}\omega^{ij}{}_\mu\delta\rho_{ij\beta\gamma} + 2\tau_{i\mu\gamma}\delta\vartheta^i{}_\beta + \rho_{ij\mu\gamma}\delta\omega^{ij}{}_\beta\Big]\Big\} + R\,,\ \delta G_2 = \frac{1}{2}\varepsilon^{0\alpha\beta\gamma}\partial_\alpha\Big[\frac{1}{2}\theta^{ij}\delta\rho_{ij\beta\gamma}\Big]\,.$$

- Going over to the notation of differential forms we get

$$\delta G = -\delta\Gamma_\infty + R\,,\quad \delta\Gamma_\infty := \oint_{S_\infty}\delta B\,, \tag{4.2a}$$

$$\delta B := (\xi\rfloor\vartheta^i)\delta H_i + \delta\vartheta^i(\xi\rfloor H_i) + \frac{1}{2}(\xi\rfloor\omega^{ij})\delta H_{ij} + \frac{1}{2}\delta\omega^{ij}(\xi\rfloor H_{ij}) + \frac{1}{2}\theta^{ij}\delta H_{ij}\,, \tag{4.2b}$$

where S_∞ is the boundary of Σ at infinity.

- If the asymptotic conditions ensure Γ_∞ to be a *finite* solution of the variational equation (6.14), the improved gauge generator

$$\tilde{G} := G + \Gamma_\infty \tag{4.3}$$

has well-defined functional derivatives. Then, the value of $\tilde{G}$ is effectively given by the value of Γ_∞, which represents the canonical *charge at infinity*.

  (a1) In the above variational equations, the variation of Γ_∞ is defined over a suitable set of asymptotic states, keeping the background configuration fixed.

- Nester and co-workers succeeded to explicitly construct a set of finite expressions Γ_∞. Although their approach yields highly reliable expressions for the conserved charges, we shall continue using the variational approach (6.14), as it can be naturally extended to a new definition of black hole entropy.

- In order to interpret black hole entropy as the canonical charge on horizon, we assume that the boundary of Σ has two components, one at spatial infinity and the other at horizon, $\partial\Sigma = S_\infty \cup S_H$.
- Now the condition of differentiability of the canonical generator G includes two boundary terms, the integrals of $\delta B = \delta B(\xi, \theta)$ over S_∞ and S_H:

$$\delta G = -\oint_{S_\infty} \delta B + \oint_{S_H} \delta B + R\,. \tag{4.4}$$

- Here, as we already know, the first term represents the asymptotic canonical charge,

$$\delta\Gamma_\infty = \int_{S_\infty} \delta B\,. \tag{4.5}$$

- The second one defines entropy S as the canonical *charge on horizon*,

$$\delta\Gamma_H := \oint_{S_H} \delta B\,. \tag{4.6}$$

(a2) The variation of Γ_H is performed by varying the parameters of a solution, but keeping surface gravity constant.

- Explicit form of entropy depends on two factors: dynamical and geometric properties of a theory and specific structure of the black hole.
- For stationary black holes in GR, the entropy formula (4.6) takes the well-known form

$$\delta\Gamma_H = T\delta S\,, \tag{4.7}$$

where $T = \kappa/2\pi$ represents the temperature and $S = \pi r_+^2$ is black hole entropy.

- The gauge generator G is regular if and only if the sum of two boundary terms vanishes,

$$\delta\Gamma_\infty - \delta\Gamma_H = 0\,, \tag{4.8}$$

which is nothing but the *first law* of black hole thermodynamics. Thus, the validity of the first law directly follows from the regularity of the original gauge generator G.
- In the framework of PG, the conserved charge is a well-established concept which has been calculated for a number of exact solutions. In contrast to that, much less is known about black hole entropy.
- We shall now test our definition of black hole entropy and the associated first law, on three illustrative examples from the family of Schwarzschild-AdS solutions, Kerr-Newmann AdS solutions and solutions with scalar hair.

- Teleparallel gravity (TG) is a subcase of PG, defined by the vanishing Riemann-Cartan curvature, $R^{ij} = 0$. Choosing the related spin connection to vanish, $\omega^{ij} = 0$, the tetrad field remains the only dynamical variable, and torsion takes the form $T^i = d\vartheta^i$. The general (parity invariant) TG Lagrangian has the form

$$L_T := a_0 T^{i\,\star}\left(a_1\,{}^{(1)}T_i + a_2\,{}^{(2)}T_i + a_3\,{}^{(3)}T_i\right) . \qquad (5.1a)$$

- In physical considerations, a special role is played by a special *one-parameter family* of TG Lagrangians, defined by a single parameter γ as

$$a_1 = 1\,, \qquad a_2 = -2\,, \qquad a_3 = -1/2 + \gamma\,. \qquad (5.1b)$$

- This family represents a viable gravitational theory for macroscopic matter, empirically indistinguishable from GR.

- Every spherically symmetric solution of GR is also a solution of the one-parameter TG. In particular, this is true for the Schwarzschild-AdS spacetime. Since ${}^{(3)}T_i = 0$, the covariant momentum H^i does not depend on γ:

$$
\begin{aligned}
H^0 &= \frac{2a_0}{r\sin(\theta)}\Big[\cos(\theta)\vartheta^1\vartheta^3 - 2N\sin(\theta)\vartheta^2\vartheta^3\Big]\,,\\
H^1 &= \frac{2a_0\cos(\theta)}{r\sin(\theta)}\vartheta^0\vartheta^3\,, \qquad H^2 = -\frac{2a_0}{r}(rN' + N)\vartheta^0\vartheta^3\,,\\
H^3 &= \frac{2a_0}{r}(rN' + N)\vartheta^0\vartheta^2\,. \qquad (5.2)
\end{aligned}
$$

- The energy of the Schwarzschild-AdS solution in TG is

$$E = m\,. \qquad (5.3)$$

- Our approach to entropy yields (integration implicit)

$$\vartheta^i{}_t\delta H_i = \big[N\delta H_0\big]_{r_+} = -16\pi a_0\big[N\delta(Nr)\big]_{r_+} = 0\,,$$
$$\vartheta^i\delta H_{it} = \big[\vartheta^2\delta H_{2t} + \vartheta^3\delta H_{3t}\big]_{r_+} = 8\pi a_0\cdot\kappa\delta(r_+^2)\,,$$

where we used $NN' = \kappa$ and $[N\delta N]_{r_+} = 0$. Thus, with $16\pi a_0 = 1$, one obtains

$$\delta\Gamma_H = T\delta S\,, \qquad S = \pi r_+^2\,. \tag{5.4a}$$

The identity $2\delta m = \kappa\delta r_+^2$ confirms the validity of the first law

$$\delta E = T\delta S\,. \tag{5.5}$$

PG-Maxwell system

- Let us extend our investigation of entropy by introducing *Maxwell field* as a matter source for gravity (PG-Maxwell system).
- The analysis is focussed on exploring thermodynamic properties of the generalized KN-AdS black hole, constructed by Baekler et al. in the late 1980s.
- Our physical system now contains also the Maxwell field characterised by the field strength $F = dA$ (2-form), where A is the electromagnetic gauge potential (1-form) and dynamical properties of the PG-Maxwell system are defined by the total Lagrangian

$$L = L_G + L_M \,, \tag{6.1}$$

where $L_M := 4a_1 \left(-\frac{1}{2} F \,{}^\star F\right)$ describes the Maxwell field interacting with gravity.

- The field equations now read

$$\delta\vartheta^i: \qquad \nabla H_i + E_i = -\tau_i\,, \tag{6.2a}$$
$$\delta\omega^{ij}: \qquad \nabla H_{ij} + E_{ij} = 0\,, \tag{6.2b}$$
$$\delta A: \qquad dH = 0\,, \tag{6.2c}$$

  where $\tau_i := \partial L_M/\partial\vartheta^i$ is the Maxwell energy-momentum current , the spin current vanishes, $\sigma_{ij} := \partial L_M/\partial\omega^{ij} = 0$ and $H := \partial L_M/\partial A = -4a_1{}^\star F$ is the electromagnetic covariant momentum.
- Asymptotic charges and entropy of a PG-Maxwell black hole are determined by the boundary integral which contains an additional contribution stemming from Maxwell field

$$\delta B_M(\xi) := (\xi\,\rfloor\,A)\delta H + (\delta A)(\xi\,\rfloor\,H)\,. \tag{6.3}$$

- δB_M is again obtained from the canonical generator.

Metric, tetrad and torsion

- The metric of a KN-AdS black hole in Boyer-Lindquist (BL) coordinates has the form

$$ds^2 = \frac{\Delta}{\rho^2}\Big(dt+\frac{a}{\alpha}\sin^2\theta d\varphi\Big)^2-\frac{\rho^2}{\Delta}dr^2-\frac{\rho^2}{f}d\theta^2-\frac{f}{\rho^2}\sin^2\theta\Big[adt+\frac{(r^2+a^2)}{\alpha}d\varphi\Big]^2\,,$$

$$\Delta(r) := (r^2+a^2)(1+\lambda r^2)-2(mr-q^2)\,, \qquad \alpha := 1-\lambda a^2\,,$$
$$\rho^2(r,\theta) := r^2+a^2\cos^2\theta\,, \qquad f(\theta) := 1-\lambda a^2\cos^2\theta\,. \qquad (6.4)$$

- Here, m, a and q are characterizing conserved charges of the solution, and $\lambda = -\Lambda/3a_0$.
- The orthonormal tetrad is chosen in the form

$$\vartheta^0 = N\Big(dt+\frac{a}{\alpha}\sin^2\theta\,d\varphi\Big)\,, \quad \vartheta^1 = \frac{dr}{N}\,, \quad \vartheta^2 = Pd\theta\,,$$
$$\vartheta^3 = \frac{\sin\theta}{P}\Big[a\,dt+\frac{(r^2+a^2)}{\alpha}d\varphi\Big]\,, \quad N = \sqrt{\Delta}/\rho\,, \quad P = \rho/\sqrt{f}\,. \qquad (6.5)$$

- The larger root of $\Delta(r) = 0$ defines the outer horizon, and the angular velocity and surface gravity have the GR form

$$\omega_+ = \frac{a\alpha}{r_+^2 + a^2}\,, \qquad \Omega_+ := \omega_+ + \lambda a = \frac{a(1+\lambda r_+^2)}{r_+^2 + a^2}\,, \tag{6.6}$$
$$\kappa = \frac{r_+^2 + 3\lambda r_+^4 + \lambda a^2 r_+^2 - a^2 - 2q^2}{2r_+(r_+^2 + a^2)}\,,\ A_H = \int_{r_+} \vartheta^2\vartheta^3 = \frac{4\pi(r_+^2 + a^2)}{\alpha}\,.$$

- For KN-AdS black holes in PG, the ansatz for torsion is formally the same as for the Kerr-AdS case

$$T^0 = T^1 = \frac{1}{N}\Big[-V_1\vartheta^0\vartheta^1 - 2V_4\vartheta^2\vartheta^3\Big] + \frac{1}{N^2}\Big[V_2\vartheta^-\vartheta^2 + V_3\vartheta^-\vartheta^3\Big]\,,$$
$$T^2 := \frac{1}{N}\Big[V_5\vartheta^-\vartheta^2 + V_4\vartheta^-\vartheta^3\Big]\,, T^3 := \frac{1}{N}\Big[-V_4\vartheta^-\vartheta^2 + V_5\vartheta^-\vartheta^3\Big], \tag{6.7}$$

where $\vartheta^- = \vartheta^0 - \vartheta^1$ and the torsion functions V_n are modified by the presence of the electric charge parameter.

Connection and curvature

- The Riemann-Cartan connection can be expressed as

$$\omega^{ij} = \tilde{\omega}^{ij} + K^{ij} \,, \tag{6.8}$$

where $\tilde{\omega}^{ij}$ is Levi-Chivita (Riemannian) connection and K^{ij} is the contortion 1-form, implicitly defined by the relation $T^i = K^i{}_k b^k$.

- The curvature 2-form $R^{ij} = d\omega^{ij} + \omega^i{}_k \omega^{kj}$ has only two nonvanishing irreducible parts:

$$^{(6)}R^{ij} = \lambda \vartheta^i \vartheta^j \,, \qquad ^{(4)}R^{Ac} = \frac{\lambda}{\Delta}(mr - q^2)\vartheta^- \vartheta^c \,. \tag{6.9}$$

The quadratic invariants (Euler, Pontryagin and Nieh-Yan) are given by

$$I_E := (1/2)\varepsilon_{ijmn} R^{ij} R^{mn} \equiv {}^\star R_{mn} R^{mn} = 12\lambda^2 \hat{\epsilon} \,,$$
$$I_P := R^{ij} R_{ij} = 0 \,, \qquad I_{NY} = T^i T_i - R_{ij} b^i b^j = 0 \,. \tag{6.10}$$

PG-Maxwell field equations

- The Maxwell potential in a KN-AdS spacetime is

$$A := -\frac{q_e r}{\rho\sqrt{\Delta}}\vartheta^0 \equiv -\frac{q_e r}{\rho^2}\left(dt + \frac{a}{\alpha}\sin^2\theta d\varphi\right), \qquad (6.11)$$

  where q_e is the electromagnetic charge parameter.
- The explicit calculation shows that basic dynamical variables $(\vartheta^i, \omega^{ij}, A)$ of a KN-AdS black hole solve the PG-Maxwell field equations if the Lagrangian parameters (a_n, b_n, Λ) and the solution parameters (λ, q, q_e) satisfy

$$\begin{aligned} &2a_1 + a_2 = 0\,, \qquad && a_0 - a_1 - \lambda(b_4 + b_6) = 0\,, \\ &3\lambda a_0 + \Lambda = 0\,, \qquad && q_e^2 = 2q^2\,. \end{aligned} \qquad (6.12)$$

- The electromagnetic charge parameter q_e differs from the metric charge parameter q. However, none of them coincides with the asymptotic Maxwell charge.

Asymptotic boundary terms

- The asymptotic values of energy and angular momentum are defined by the boundary term $\delta B(\xi)$.
- Let us mention that Carter and Henneaux and Teitelboim demonstrated that the asymptotic metric of Kerr-AdS spacetimes cannot be properly described in BL coordinates. They found a new set of coordinates in which this deficiency is brought under control. However, our variational approach allows a simpler procedure, in which only the subset (t, φ) of the BL coordinates is transformed

$$T = t\,, \qquad \phi = \varphi - \lambda a t\,. \tag{6.13a}$$

- Consequently. the components of metric transform as

$$g_{T\varphi} = g_{t\varphi} + g_{\varphi\varphi}\,, \quad \Omega_+ := \left(\frac{g_{T\varphi}}{g_{\varphi\varphi}}\right)_{r_+} = \omega_+ + \lambda a \tag{6.13b}$$

Angular momentum and energy

- It is interesting to note that the contribution of the Maxwell field in the expression $\delta B(\xi)$, yields vanishing boundary terms at infinity, but not at horizon.
- The angular momentum is defined by $\delta E_\varphi := \delta\Gamma_\infty(\partial_\varphi)$. By summing up the nonvanishing contributions one obtains

$$\delta E_\varphi = 16\pi a_1 \delta\left(\frac{ma}{\alpha^2}\right). \tag{6.14}$$

- Going over to energy, we obtain

$$\delta E_t = 16\pi a_1 \left[\frac{m}{2}\delta\left(\frac{1}{\alpha}\right) + \delta\left(\frac{m}{\alpha}\right)\right].$$

- The result is not δ-integrable but, as we mentioned above, it can be corrected by moving to the untwisted coordinates

$$\delta E_T = \delta E_t + \lambda a \delta E_\varphi = 16\pi a_1 \delta\left(\frac{m}{\alpha^2}\right). \tag{6.15}$$

Entropy

- First we analyse the PG part of the boundary term at horizon, $\delta\Gamma_H$, where the Killing vector ξ is given by

$$\xi := \partial_T - \Omega_+\partial_\phi = \partial_t - \omega_+\partial_\varphi \,. \tag{6.16}$$

- This part defines the black hole entropy. After very lengthy calculation we get

$$\begin{aligned} (\delta\Gamma_H)^{PG} &= 8\pi a_1 \kappa\delta\left(\frac{r_+^2 + a^2}{\alpha}\right) = T\delta S\,, \\ S &:= 16\pi a_1 \frac{\pi(r_+^2 + a^2)}{\alpha}\,, \end{aligned} \tag{6.17}$$

  where $T := \kappa/2\pi$ is the temperature.
- Thus, entropy is as the conserved charges proportional to the GR value.

Maxwell boundary term and the first law

- The asymptotic electric charge Q can be defined by

$$Q = -\int_{S_\infty} H = 4a_1 \int_{S_\infty} \frac{q_e}{\rho^4}(r^2 - a^2\cos^2\theta) b^2 b^3 = 16\pi a_1 \frac{q_e}{\alpha}\,. \tag{6.18}$$

- The electric potential Φ is defined by

$$\Phi := A_\xi\Big|_{r_+}^{\infty} = -\frac{q_e r_+}{\rho_+^2 N} b^0{}_\xi\Big|_{r_+}^{\infty} = \frac{q_e r_+}{r_+^2 + a^2}\,. \tag{6.19}$$

- Then, the Maxwell contribution on horizon has the form

$$(\delta\Gamma_H)^M = A_\xi \delta H + (\delta A) H_\xi = A_\xi \delta H = \Phi\,\delta Q\,. \tag{6.20}$$

- Combining this relation with the already obtained results, one can immediately conclude that the first law $\delta\Gamma_H = \delta\Gamma_\infty$ takes the form

$$T\delta S + \Phi\delta Q = \delta E_T - \Omega_+ \delta E_\varphi\,. \tag{6.21}$$

Dynamics and boundary terms

- We shall now extend the investigations to hairy black holes. Our attention is focussed on the Martinez-Teitelboim-Zaneli (MTZ) solution of GR as a characteristic representative of the family of Riemannian black holes with scalar hair.
- The Lagrangian L (4-form) describes the scalar matter coupled to the gravitational field,

$$L = L_G + L_M \,, \qquad (7.1)$$
$$L_M := \frac{1}{2} d\phi({}^\star d\phi) + {}^\star V \,,$$

  where $V = V(\phi)$ is a self-interaction potential.
- The covariant momentum associated to the scalar fiel reads

$$H_\phi := \frac{\partial L_M}{\partial d\phi} \,, \qquad (7.2)$$

- The field equation can be written in a compact form

$$\begin{aligned} \delta\vartheta^i \quad &: \quad \nabla H_i + E_i = -\tau_i\,, && (7.3a)\\ \delta\omega^{ij} \quad &: \quad \nabla H_{ij} - (b_i H_j - b_j H_i) = 0\,, && (7.3b)\\ \delta\phi \quad &: \quad -dH_\phi + \partial_\phi{}^\star V = 0\,. && (7.3c) \end{aligned}$$

  Here, $E_i := \partial L_G/\partial\vartheta^i$ and $\tau_i := \partial L_M/\partial\vartheta^i$ are the gravitational and matter energy-momentum currents .

- The Hamiltonian approach to black hole thermodynamics is based on the already mentioned ideas. The related boundary integral contains a contribution related to a scalar field

$$\delta B_M(\xi) \quad := \quad -(\delta\phi)(\xi \rfloor H_\phi)\,. \qquad (7.4)$$

- For static black holes, the Killing vector ξ has the form $\xi = \partial_t$.

The MTZ black hole

- In the Riemannian limit $T_i = 0$, hence $H_i = 0$, the equation (7.3b) takes the form $\nabla H_{ij} = 0$, which is, *generically*, not satisfied by the MTZ spacetime; the MTZ black hole is a solution of PG only in two complementary subcases:
  (c1) in GR, where $T^i = 0$ and moreover, $b_n = 0$;
  (c2) in the teleparallel gravity, where $R^{ij} = 0$.
- Riemannian geometry of the MTZ black hole is determined by the metric

$$ds^2 = C^2\left(N^2 dt^2 - \frac{dr^2}{N^2} - r^2 d\sigma^2\right), \quad C^2 := \frac{r(r+2Gm)}{(r+Gm)^2}\,,$$
$$N^2 := \frac{r^2}{\ell^2} - \left(1 + \frac{Gm}{r}\right)^2, \tag{7.5}$$

where $d\sigma^2$ is the metric of a 2-dimensional manifold Σ with constant negative curvature, rescaled to -1.

- The manifold Σ is locally isometric to the hyperbolic manifold H^2, the metric of which can be written in the form

$$d\sigma^2 = d\rho^2 + \sinh^2\rho\, d\varphi^2\,, \tag{7.6}$$

  where $\rho \in [0, \infty)$ and $\varphi \in [0, 2\pi)$.
- Since H^2 has infinite area, which is a serious obstacle in thermodynamic considerations, Σ is chosen to be of the form $\Sigma = H^2/\Gamma$, where Γ is a discrete subgroup of $SO(1, 2)$, the isometry group of H^2. This choice ensures the area of Σ to be finite.
- The asymptotic form of N^2 suggests that the parameter m is a measure of the black hole mass, whereas its zeros determine the event horizon:

$$r_+ = \frac{\ell}{2}\left(1 + \sqrt{1 + \frac{4Gm}{\ell}}\right)\,. \tag{7.7}$$

- We choose the orthonormal tetrad as

$$\vartheta^0 = A dt\,, \quad \vartheta^1 = \frac{dr}{B}\,, \quad \vartheta^2 = D d\rho\,, \quad \vartheta^3 = D \sinh\rho\, d\varphi\,,$$
$$A := CN, \quad B := N/C, \quad D := Cr\,. \tag{7.8}$$

- The horizon area takes the form

$$A_H = \int_H \vartheta^2\vartheta^3 = \ell(2r_+ - \ell)\sigma\,, \quad \sigma := \int_H d\rho \sinh\rho\, d\varphi\,,$$

while surface gravity and the black hole temperature are

$$\kappa := B\partial_r A\big|_{r_+} = \frac{1}{\ell}\sqrt{1 + \frac{4Gm}{\ell}} = \frac{1}{\ell^2}(2r_+ - \ell)\,, \quad T := \frac{\kappa}{2\pi}\,.$$

- The Riemannian spin connection reads

$$\omega^{01} = -\frac{A'}{A} B\,\vartheta^0\,, \quad \omega^{1c} = \frac{D'}{D} B\,\vartheta^c\,, \quad \omega^{23} = \frac{\cosh\theta}{D\sinh\theta}\,\vartheta^3\,, \tag{7.9}$$

where $c = (2, 3)$.

- Adopting the notation $\mu := Gm$, the solution of the scalar field equation reads

$$\phi := \sqrt{k}\,\text{atanh}\left(\frac{\mu}{r+\mu}\right), \qquad V(\phi) := +\frac{k}{\ell^2}\sinh^2\left(\frac{\phi}{\sqrt{k}}\right). \qquad (7.10)$$

- The value of the normalization constant k is fixed by gravitational field equations

$$\Lambda + \frac{3a_0}{\ell^2} = 0\,, \qquad k = 12a_0 = \frac{3}{4\pi}\,. \qquad (7.11)$$

- When $H_i = 0$, the general boundary term is reduced to

$$\delta B(\xi) = \frac{1}{2}(\xi \rfloor \omega^{ij})\delta H_{ij} + \frac{1}{2}\delta\omega^{ij}(\xi \rfloor \delta H_{ij}) - (\delta\phi)(\xi \rfloor H_\phi)\,, \qquad (7.12)$$

  where $\xi = \partial_t$. Hence, energy and entropy of the MTZ black hole are determined by calculating the corresponding boundary integrals $\delta\Gamma_\infty$ and $\delta\Gamma_H$, respectively.

Energy

- The gravitational and scalar field contributions to the boundary term are

$$\begin{aligned}\delta B_G &= 12a_0\left(-\frac{\mu r}{\ell^2}+\frac{4\mu^2}{\ell^2}\right)\sigma\delta\mu+4a_0\sigma\delta\mu\,,\\ \delta B_\phi &= k\delta\mu\Big(\frac{\mu r}{\ell^2}-\frac{4\mu^2}{\ell^2}\Big)\sigma\,,\end{aligned}$$

Both terms are divergent, but for $4a_0 = 3/4\pi$, their sum yields the finite expression for energy:

$$\delta\Gamma_\infty := \delta B_G+\delta B_\phi=\frac{\delta\mu}{4\pi}\sigma \quad\Rightarrow\quad \delta E=\frac{\sigma}{4\pi}\delta\mu\,. \tag{7.14}$$

Entropy and the first law

- Since δB_ϕ is proportional to the tetrad function B, it vanishes on horizon. Hence, the only nontrivial contribution comes from δB_G,

$$\delta\Gamma_H = \int \delta B_G(r_+) = \frac{\sigma}{4\pi}\delta\mu\,. \tag{7.15}$$

- Using the explicit expressions for A_H and T, one can derive the identity

$$\delta\Gamma_H = T\delta S\,, \qquad S := \frac{A_H}{4\pi}\,, \tag{7.16}$$

which identifies S as entropy.

- These results imply the validity of the first law,

$$\delta\Gamma_\infty = \delta\Gamma_H \quad \Rightarrow \quad \delta E = T\delta S\,. \tag{7.17}$$

- We investigated the notion of entropy in the general four-dimensional PG. Our approach was based on the idea that black hole entropy can be interpreted as the conserved charge on horizon.
- We constructed the canonical generator G of gauge symmetries as an integral over the spatial section Σ of spacetime, which has to be a regular (differentiable) functional on the phase space. The regularity can be ensured by adding to G a suitable surface term Γ_∞ defined on the boundary of Σ at infinity.
- The form of Γ_∞ is determined by the variational equation and its value defines the asymptotic charge.
- For a black hole solution, Σ has two boundaries, one at infinity and the other at horizon. The condition of regularity of G includes two boundary terms, Γ_∞ and Γ_H.

- The new boundary term Γ_H, defines entropy as the canonical charge on horizon. The regularity of G represents just the first law of black hole thermodynamics.
- For a number of black holes in PG:
  - Riemannian Schwarzschild-AdS (SAdS)
  - Schwarzschild-AdS solutions with torsion (Baeckler) solution
  - Teleparallel Schwarzschild- AdS
  - Kerr
  - Kerr-AdS black holes *with or without torsion*,

  it was found, somewhat unexpectedly, that entropy retains its area form, up to multiplicative factor. Also the first law retains the standard form.
- The analysis of the Kerr-Newmann solution led also to the expected results as well as the investigation of the black holes with scalar hair.