Remarks on two ”solvable” ncQFT models

Quantum + Fuzzy:
Workshop in honour of the birthday of
Professor Maja Buric

Harald Grosse

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Happy Birthday Maja

- Congratulations
- many contributions
- many publications
- many coworker
- a nice atmosphere
- Fuzzy black hole, fuzzy de Sitter
- Fuzzy cosmology
- ....

Many collaborations Maja +...

- John, Voja
- Latas, Zoupanos
- Trampetic, Nenadovic
- Prekrat
- more than 20 more
- renormalization questions
  Maja + Voja ...
- Maja + M. Wohlgenannt
  !!! curvature!!!
- Maja + HG + John :
  Gauge model...

Quartic Model

- Φ^4 on R^4_Θ nonren.!
  add box term...ren.!
- symmetric coupling leads
  to a matrix model
- $S = TrE\Phi^2 + \lambda Tr\Phi^4$
- $\beta_\lambda = 0$
- nonperturbative ren.
- 2 Ren-const Z, m^2
- WI+SD Vincent,...
  solvable, NONTRIVIAL!
- blobbed top. rec.

review: J Brahnal+HG+A Hock + R Wulkenhaar 2110.11523

Deform Euclidean $\mathbb{R}^4$

Formulation

ϕ^4 on nc $\mathbb{R}^4$, $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$ antisymmetric, or star product

$$(a * b)(x) = \int dy \int dk a(x + \frac{\theta k}{2}) b(x + y) e^{iky}$$

ϕ^4 action

$$S = \int dp(p^2 + m^2)\phi_p\phi_{-p} + \lambda \int \prod_{j=1}^{4} (dp_j \phi_{p_j})\, \delta(\sum_{j=1}^{4} p_j) e^{-i\sum_{i<j} p_i \theta p_j}$$

Ribbon graphs, not ren., implied by IR/UV mixing

Possible solution: modify action

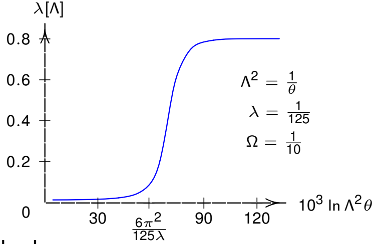
Theorem: H. G. + R. Wulkenhaar

Theorem: Action is pert. and nonpert. ren

$$S = \int d^4x \Big(\frac{1}{2}\partial_\mu\phi\star\partial^\mu\phi + \frac{\Omega^2}{2}(\tilde{x}_\mu\phi)\star(\tilde{x}^\mu\phi) + \frac{\mu_0^2}{2}\phi\star\phi + \lambda\phi\star\phi\star\phi\star\phi\Big)(x)$$

$\beta_\lambda = 0$ at $\Omega = 1$,
flow bounded, L. ghost killed!
use SD equation and WT-I

$\lambda[\Lambda]$ diverges in comm. case



- perturbation theory valid at all scales!
- non-perturbative construction "done"!

Solution $\Phi^4_\ominus$: SD and WT identities

Regard the planar two-point function G_{ab} as evaluation of a holomorphic function $G(\zeta, \eta)$ (for genus zero): (*)

$$(\zeta + \eta + 2c)ZG(\zeta, \eta) = 1 - \frac{\lambda}{N} \sum_{k=1}^{N} \Big(Z^2 G(\zeta, \eta)\, G(\zeta, E_k) - \frac{ZG(E_k, \eta) - ZG(\zeta, \eta)}{E_k - \zeta} \Big)$$

Non-linear equ., hard to solve. Solved in $D = 2$ E. Panzer + R. W,
Divergence problem: adjust $Z(N, \lambda)$ and $c(N, \lambda)$ to achieve limit $N \to \infty$.
Start from a holomorphic function J (to be determined) and define for

appropriately chosen pre-images $J(\varepsilon_k) = E_k$ another holomorphic function $\mathcal{G}$ by $\mathcal{G}(z, w) = G(J(z), J(w))$.

The key step is to relate J and $\mathcal{G}$ by

$$J(z) + 2c + \frac{\lambda}{N} \sum_{k=1}^{N} Z\mathcal{G}(z, \varepsilon_k) + \frac{\lambda}{N} \sum_{k=1}^{N} \frac{1}{J(\varepsilon_k) - J(z)} = -J(-z).$$

$$(J(w) - J(-z))Z\mathcal{G}(z, w) = 1 + \frac{\lambda}{N} \sum_{k=1}^{N} \frac{Z\mathcal{G}(\varepsilon_k, w)}{J(\varepsilon_k) - J(z)},$$

is a *linear equ.!*

Solution $\Phi^4_\ominus$, finite dof

When N is finite and fixed, J is a rational function with $N+1$ pre-images:

Theorem

Construct 2N functions $\{\varepsilon_k(\lambda), \varrho_k(\lambda)\}_{k=1,\dots,N}$ *as implicitly defined solution of the system*

$$E_k = J(\varepsilon_k), \qquad \frac{1}{\varrho_k} = J'(\varepsilon_k), \qquad \textit{where} \quad J(z) = z - \frac{\lambda}{\mathcal{N}} \sum_{j=1}^{d} \frac{\varrho_j}{\varepsilon_j + z}$$

and $\lim_{\lambda\to 0}(\varepsilon_k, \varrho_k) = (E_k, 1)$. *(*) is solved for* $c = 0$ *and* $Z = 1$ *by* $G(\zeta,\eta) = \mathcal{G}(J^{-1}(\zeta), J^{-1}(\eta))$, *where* $\mathcal{G}$ *is the rational (and symmetric) function*

$$\mathcal{G}(z,w) = \frac{1 - \dfrac{\lambda}{N}\displaystyle\sum_{k=1}^{N} \frac{1}{(J(z)-J(\varepsilon_k))(J(\varepsilon_k)-J(-w))} \prod_{j=1}^{N} \frac{J(w)-J(-\widehat{\varepsilon_k}^j)}{J(w)-J(\varepsilon_j)}}{J(w)-J(-z)}.$$

Here, $z \in \{u, \hat{u}^1, \dots, \hat{u}^N\}$ *is the list of roots of* $J(z) = J(u)$.

Solution Φ^4_Θ, N = ∞

Use Cauchy residue th., Lagrange inversion th., Bürmann formula

$G_{a,b} = \frac{e^{N_{a,b}}}{a+b+\mu^2}$

$$N_{a,b} = \int_{-\infty}^{\infty} \frac{dt}{2i\pi} log(a - J(-\mu^2/2 - it)) \frac{d}{dt} log(b - J(-\mu^2/2 + it))$$
$$\text{- log(a - (-}\mu^2/2 - it)) \frac{d}{dt} log(b - (-\mu^2/2 + it))$$
$$\text{- log(- J(-}\mu^2/2 - it)) \frac{d}{dt} log(-J(-\mu^2/2 + it))$$
$$\text{+ log(- (-}\mu^2/2 - it)) \frac{d}{dt} log(-(-\mu^2/2 + it))$$

All higher correlation fcts. $G_{a,b,c,d} = \frac{G_{a,b}G_{c,d} - G_{b,c}G_{a,d},}{(a-c)(b-d)}$

(for genus zero) are weighted polynomials of $G_{a,b}$
Higher genus functions follow blobbed topological recursion (proven for genus zero and one)

Two independent dimensions

1. Topological dim. 2 from expansion into ribbon graphs
   - dual to triangulations (Φ^3) or quadrangulations (Φ^4) of 2D-surfaces
   - partition function counts them = 2D quantum gravity
   - non-planar ribbon graphs suppressed in large-$\mathcal{N}$ limit
2. Dynamical dimension D encoded in spectrum of the operator E,

$$D = \inf\{p \in \mathbb{R}_+ \ : \ \mathrm{tr}((1+E)^{-\frac{p}{2}}) < \infty\}$$

   - ignored in 2D quantum gravity...
   - highly relevant for renormalisation of matricial QFT

polynomial	finite	super-ren	just ren.	not ren.
Φ^3	$D < 2$	$2[\frac{D}{2}] \in \{2,4\}$	$2[\frac{D}{2}] = 6$	$2[\frac{D}{2}] > 6$
Φ^4	$D < 2$	$2[\frac{D}{2}] = 2$	$2[\frac{D}{2}] = 4$	$2[\frac{D}{2}] > 4$

NON-TRIVIAL

Triviality of $\Phi^4_{4+\epsilon}$ Aizenman, Duminil-Copin ; Fröhlich

- lattice model gives mean field exponents -free fields
- use correlation inequalities and scaling of lattice constant

CLAIM: Φ^4_Θ + oscillator is a NON-TRIVIAL MODEL

- dimension drop occurs H G,A Hock,R W

$$J_D(x) = x - \lambda(-x)^{(D/2)} \int_0^{\Lambda_D^2} dt \frac{\rho_\lambda(t)}{(t+1)^{(D/2)}(t+1+x)}$$

$$\rho_\lambda(x) = \rho_0(J_D(x))$$

$$J_4(z) = z.{}_2F_1(\alpha_\lambda, 1-\alpha_\lambda, 2; -\frac{-z}{\mu^2})$$

$$\alpha_\lambda = \frac{arcsin[\lambda\pi]}{\pi}$$

Asymptotic behavior !

- ${}_2F_1$ behaves for large z as $\frac{1}{z^{\alpha}\lambda}$
- Moyal space has spectral dimension: $4 - 2\frac{arcsin(\lambda\pi)}{\pi}$ for $|\lambda| < \frac{1}{\pi}$

Deform Minkowski, def. Wedge local fields

H G, Lechner; (Buchholz and Summers)

Quantum fields over deformed Minkowski space time

NC coordinates: $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$, $[\hat{x}^\mu, \theta^{\sigma\tau}] = 0$

Field operators are defined as tensor product:

$$\Phi^{\otimes}(x) = \int d\mu_p (e^{ipx} e^{ip\hat{x}} \otimes \Phi_p)$$

Vacuum states: take: $\omega_\theta = \nu \otimes < \Omega, .\Omega >$ ω_θ is independent of ν

GNS rep. of pol algebra build from $\Phi^{\otimes}(f)$ wrt ω_θ

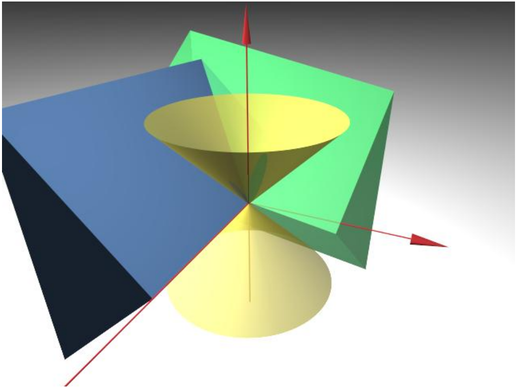
- There exists a unitary map from $\Phi^{\otimes}(f)$ to ϕ_Θ acting on Hilbert space H of the undeformed theory $\Rightarrow$ cov. prop. of ϕ^θ w.r.t. undeformed rep. U
- Consider family of fields

Wedges

We relate the antisymmetric matrices to Wedges: $W_1 = \left\{ x \in \mathbb{R}^D | x_1 > |x_0| \right\}$

act on standard wedge by proper Lorentz transformations $i_\Lambda(W) = \Lambda W$.

Stabilizer group $SO(1,1)xSO(2)$, corresponds to boosts and rotations.



Wedges and Wedge local QF

$\mathcal{A} = \{\gamma_\Lambda(\theta_1) | \Lambda \in \mathcal{L}_+\}$, $\quad \theta(\Lambda W_1) := \gamma_\Lambda(\theta_1) = \Lambda\theta_1\Lambda^\dagger$

Define wedge local fields through: $\phi = \{\phi_W | W \subset \mathcal{W}_0\}$ Lorentz transf. of W_1 get family of fields, covariance and localization in wedges.

$$U_{y,\Lambda}\Phi_W(x)U^\dagger_{y,\Lambda} = \Phi_{\Lambda W}(\Lambda x + y)$$

Theorem

Let $\kappa_e \geq 0$ the family $\Phi_W(x)$ is a wedge local quantum field:

$$[\phi_{W_1}(f), \phi_{-W_1}(g)](\psi) = 0,$$

for $supp(f) \subset W_1$, $supp(g) \subset -W_1$.

Proof relies on spectrum condition and support properties

Relative Entropy

$$\Phi_\Theta(f) = \int \frac{d^3p}{\omega_p}(f_p^- e^{-ip\Theta P} a_p + f_p^+ e^{ip\Theta P} a_p^\dagger) = \int d\mu(p)(f_p^- A_p + f_p^+ A_p^\dagger))$$

f^+, f^- on-shell! $A_p, A_p^\dagger$ obey def. Algebra...

Compare two states $|\Omega>$ and $e^{i\Phi_\theta(f)}|\Omega>$ coherent state for $\Theta = 0$

Use Tomita-Takesaki modular theory

- $\Delta = e^{2\pi L_{01}}$ L_{01} is boost operator (B-W th) $L_{01} = \int d^3x x^1 T_{00}(x)$

Use ARAKI UHLMANN equation

- $S(\omega‘, \omega) = i\frac{d}{dt}|_{t=0} < \Omega|U\Delta^{it}U^\dagger\Delta^{-it}\Omega >=$

$$= -2\,\pi < \Omega|e^{i\Phi_\Theta(f)}L_{01}e^{-i\Phi_\Theta(f)}\Omega >$$

Calculate quantum corrections to relative entropy

$$L_{01} = \int d\mu(k) a_k^\dagger \omega_k \frac{d}{dk^1} a_k$$

$$S(\omega‘, \omega) = S_0(\omega‘, \omega) + \frac{8\pi}{3}\Theta(\int d^3k f_k^+ f_k^-)^2$$

gives modified Bekenstein bound HG + A. Much + R. Verch (tbp)

Summary

DEFORMED EUCLIDEAN SPACE-TIME

- Found renormalizable ncQFT for matter fields
- β - function vanishes identically
- constructive procedure (partly) done
- two point fct known: NON-TRIVIAL integrable?

  J Brahnal+H G+A Hock+R W 2110.11789
- partition fct.s: solve Pde.s HG, + A. Sako, Tanomata, + R W 2311.10974,2308.11523

DEFORMED MINKOWSKI Space-Time

- Found Wedge local quantum fields
- Calculated quantum corrections to relative entropy (tbp)

Congratulations

to **MAJA**

Congratulations to your birthday