Quantum Spacetime, Noncommutative Geometry and Observers

Fedele Lizzi

Università di Napoli *Federico II*

INFN Napoli

Institut de Ciencies del Cosmos, Universitat de Barcelona

includes collaborations and discussions with M. Dimitrijevic Ciric, G. Fabiano, G. Fiore, G. Gubitosi, N. Konjik, M. Kurkov, M. Manfredonia, F. Mercati, T. Poulain, J. Relancio, L. Scala and P. Vitale

Maja65 2024

In the past century we have seen more and more objects of physical interest have become “quantum”.

First the energy levels, in a black body and in atoms, then particle motions, then fields, gravity and now computing, and information, and if you look deep in the web: quantum astrology, quantum healing and medicine, and a dishwasher detergent.

Lamentably for your health, predictions of the future and shine of your dishes I will concentrate on two of the many quantum beasts:

Quantum spacetime

Quantum Observers/Reference Frames

Space, and spacetime, which have been the background arena of all physical theories, become the protagonist in general relativity a theory of curved spacetime. The dynamical variable of the theory is spacetime itself. Curvature is described by a matrix, the metric, which in the end is a field, and I may think of quantizing is as a (spin 2) field, but this attempt has not been successful. And I hasten to add that it is in good company, no attempt has been successful.

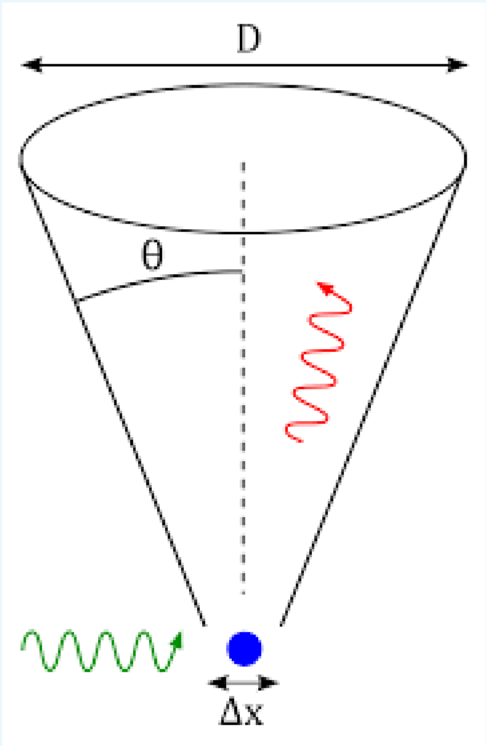
In the following I will take the point of view of Noncommutative geometry. The idea is that the object to quantize is spacetime itself, giving thus rise to a Quantum Spacetime. I will concentrate on kinematics, describing the space by a noncommutative algebra, which can be sometimes described by noncommuting coordinate functions.

There are many ways to interpret NCG, I will describe in this talk one based on symmetries leading to κ-Minkowski space, but there are other points of view, notably the original one proposed by Alain Connes based on spectral properties of algebras and operators.

The Noncommutative Geometry programme has already been successful once: Quantum Mechanics.

Classical Mechanics takes place in phase space, i.e. the space whose points are all possible positions and velocities When there are not magnetic fields.

A heuristic way to see that something must be done in phase space, which cannot be made of usual points, is the Heisenberg microscope. Which uses special relativity (and the fundamental speed of light $\boxed{c}$), the quantum of action $\boxed{\hbar}$, and the fact that light is made of photons, of energy inversely proportional to the frequency



This is the heuristic way, fortunately we have a full theory, based on noncommuting operators which works rather well.

I dare not affirm that quantum mechanics is well understood, but I venture to say that we have a solid mathematical framework with which we can work.

So far we had used only two of the fundamental constants: speed of light $\boxed{c}$, and Planck's constant $\boxed{\hbar}$.

If we attempt to define points in space(time) at very short distance we run into trouble if we put together quantum mechanics and gravity.

There is a phenomenon noticed for the first time by Bronstein in 1938, but presented independently in a modern and most terse way by Doplicher, Fredenhagen and Roberts in 1994.

I will present a caricature of these arguments, which however captures the main idea in a nontechnical way.

It is a variant of the Heisenberg microscope described above.

We are interested only is space, and not momentum, for which there is no limitation in quantum mechanics to an arbitrary precise measurement of $\boxed{x}$

Including gravity there is a length scale obtained combining the $\boxed{c}$, $\boxed{\hbar}$ and Newton's constant: $\boxed{\ell = \sqrt{\frac{\hbar G}{c^3}} \simeq 10^{-33}\text{cm}}$

In order to "measure" the position of an object, and hence the "point" in space, one has to use a very small probe, which has to be very energetic, but on the other side general relativity tells us that, if too much energy is concentrated in a small region, a black hole is formed.

It is possible (ideally) to detect the BH, but not to "see" anything inside its horizon. Again there is a limit to the precision of the measurement.

For a rigorous statement we would need a full theory of **quantum gravity**. A theory which do not (hopefully yet) posses.

We can try a replica of what was done for quantum mechanics, and consider a space in the commutator between the commutator among the coordinates is constant. Sometimes it is called DFR (Doplicher, Fredenhagen, Robers) non-commutativity, or Moyal or Gronëwold-Moyal, who introduced the deformed product which generalises this kind of noncommutativity. It also featured in the famous article of Seiberg and Witten on noncommutative geometry and strings.

$$\boxed{[x^\mu, x^\nu] = \mathrm{i}\theta^{\mu\nu}}$$

This is a spacetime replica of the quantum phase space canonical commutation relations, with $\boxed{\hbar}$ substituted by $\boxed{\theta}$. This meant that we could use all the experience and technology acquired for quantum mechanics.

The problem is that the commutation relation breaks Lorentz invariance, choosing two directions is space, a vector and a pseudovector.

We have a problem with symmetries

In parallel with the development of Noncommutative Geometry there has been the theory of Quantum Groups and Hopf Algebras.

Usually we describe symmetries via transformations described by Lie Algebras and Groups, and their representations. Both are characterized by a product, in the first case a Lie Bracket, non associative but which respects the Jacobi identity. In the second the usual group product, which is associative but in general noncommutative. The two structures are related by an exponentiation map.

A Hopf Algebra has additional structures in additions to being an algebra, i.e. a set with two operations and some other properties, think for example at a Lie algebra or the set of functions on some manifold. The latter is associative, the former is not (but it has Jacobi).

While a product is a map from two elements of the algebra into a single one, the coproduct tells how to put together representations, formally it is a map from one copy of the algebra into the tensor product of two copies. Something like

$$\boxed{\Delta(f) = f \otimes 1 + 1 \otimes f}$$

which can be seen as a rendition of the Leibnitz rule if the Lie algebra is represented as differential operators

or

$$\boxed{\Delta(f) = f \otimes f}$$

which instead is relevant for the case of functions on a group for which $\boxed{f(gg') = f(g)f(g')}$

Two more structures, counit and antipode, duals of the unity and the inverse are important but not relevant for this seminar.

Deformations of these structures, either the coproduct, of the product, in a controlled way, leads to quantum groups.

I have not time to go into details, and I will just use some properties without detailing how they emerge. The mathematical theory is quite sophisticated.

Quantum symmetries can be symmetries of quantum spacetime

For $\boxed{\kappa}$-Minkowski described below in fact it worked the other way around, first the $\boxed{\kappa}$-Poincaré was introduced, then the space is found as homogenous space.

I will consider a particular class of quantum spacetimes: that described by a noncommutative geometry described by noncommutative coordinates, of the Lie-algebra kind.

These spaces break Poincaré symmetry, but they still enjoy a quantum symmetry

Indeed in some cases the symmetry preceded the space. The symmetries will be Hopf algebras/quantum groups.

While the examples will be specific, I believe that the issues I will touch, especially the need to have quantum observers, should be a general feature of a quantum spacetime

I will therefore consider in the following a particular class of noncommutative geometries described by noncommuting variables, something of the kind

$$\boxed{[x^\mu, x^\nu] = \Theta^{\mu\nu}(x)}$$

A space defined by noncommuting coordinates is, by all means, not the only form of quantum spacetime.

I will concentrate on particular definite cases, for which $\boxed{\Theta}$ is either constant, the DFR case discussed above, and will call this $\boxed{\theta}$-Poincaré, or linearly depends on the coordinates, such as the very famous $\boxed{\kappa}$-Minkowski

$$\boxed{[x^0, x^i] = \mathrm{i}\ell x^i \quad [x^i, x^j] = 0}$$

I will use the notation $\boxed{\ell = \kappa^{-1}}$ to have the deformation parameter a small length.

Another case I will discuss, developed together with the Serbian school, if there is time, is angular noncommutativity, which I will call $\boxed{\varrho}$-Minkowski.

$$\boxed{[x^0, x^1] = -\mathrm{i}\varrho x^2 \quad [x^0, x^2] = \mathrm{i}\varrho x^1 \quad \text{all other commutators being zero.}}$$

a variant of this (which I will not discuss) would be and a purely spatial version of the above, which we call $\boxed{\lambda}$-Minkowski.

$$\boxed{[x^3, x^1] = -\mathrm{i}\lambda x^2 \quad [x^3, x^2] = \mathrm{i}\lambda x^1 \quad \text{all other commutators being zero.}}$$

Why do we need symmetries? One of the reasons, probably the main one, is the different observers, sitting in different reference frames are related by symmetry transformations.

In particular unitary operators which are a representation of the group.

I will make an attempt to use (a small part) of these quantum symmetries, relating them to reference frames and observers.

I will use the terms reference frame and observer in an interchangeable way as synonyms. Usually the general relativity community uses the first, and quantum people the second. The two concepts are difficult to define, and depending on the definition, may not be the same. This is a very interesting phylosophical issue, but I will not go into this. For me an observer defines a reference frame, the one in which she operates.

I will study this a noncommutative geometry using the usual techniques of quantum mechanics. Let me first briefly recall a well known case study: Quantum Phase Space of a particle.

Phase space is a six-dimensional space spanned by $\boxed{(q^i, p_i)}$. Quantization introduces the commutation relation $\boxed{[q^i, p_j] = \mathrm{i}\hbar\delta^i_j \ ,}$

The most common representations of position and momenta is operators on $\boxed{L^2(\mathbb{R}^3_q)}$

$$\boxed{\widehat{q}^i\psi(q) = q^i\psi(q) \ ; \qquad \widehat{p}_i\psi(q) = -\mathrm{i}\hbar\frac{\partial}{\partial q^i}\psi(q) \ .}$$

$\boxed{\widehat{q}}$'s and $\boxed{\widehat{p}}$'s are unbounded selfadjoint operators with a dense domain. The spectrum is the real line (for each $\boxed{i}$).

They have no eigenvectors but improper eigenfunctions: distributions.

Since the $\widehat{q}^i$'s commute it is possible to have a simultaneous improper eigenvector of all of them, these are the Dirac distributions $\boxed{\delta(q-\bar{q})}$ for a particular $\boxed{\bar{q}}$ vector in $\boxed{\mathbb{R}^3}$ For a particular momentum $\boxed{\bar{p}}$ the improper eigenfunctions of the $\boxed{\widehat{p}_i}$ are plane waves $\boxed{\mathrm{e}^{\mathrm{i}\bar{p}_i q^i}}$.

Formally, the eigenvalue equation $\boxed{\partial_q \psi(q) = \alpha\psi(q)\ ,\alpha \in \mathbb{C}^3}$ is solved by $\boxed{\mathrm{e}^{\alpha\cdot q}}$ with a vector $\boxed{\alpha}$

No function of this kind is square integrable, there are no (proper) eigenfunctions. The operator $\boxed{\widehat{p}}$ is self-adjoint on the domain of absolutely continuous functions. $\boxed{\alpha}$ must be pure imaginary because the distributions must be well defined on the domain of selfadjointness of the operators.

The improper eigenfunctions of momentum are physically interpreted as infinite plane waves of precise frequency

Implicitly we have chosen $\boxed{\widehat{q}^i}$ as a complete set of observables, the description of a quantum state as a function of positions. $\boxed{|\psi(q)|^2}$ (normalized) is the density probability to find the particle at position $\boxed{q}$.

The $\boxed{\psi}$ is complex and contains also the information about the density probability of the momentum operator.

We could have chosen $\boxed{\widehat{p}}$ as complete set. Then we would have the Fourier transformed $\boxed{\phi(p)}$. It is important that the Fourier transform is an isometry, it maps normalized functions of positions into normalized functions of momenta.

And we have other choices for complete sets, number operators, angular momentum ...

I now want to reproduce this discussion first for $\boxed{\kappa}$-Minkowski.

This is a quantum space, but I will only consider its kinematic, and leave $\boxed{\hbar}$ alone.

Look for a representation of the $\boxed{x^\mu}$ on $\boxed{L^2(\mathbb{R}^3)}$:

$$\boxed{\hat{x}^i\psi(x) = x^i\psi(x)}$$

$$\boxed{\hat{x}^0\psi(x) = \mathrm{i}\ell\left(\sum_i x^i\partial_{x^i} + \frac{3}{2}\right)\psi(x) = \mathrm{i}\ell\left(r\partial_r + \frac{3}{2}\right)\psi(x).}$$

Positions are multiplicative operators, time is dilation. The $\boxed{3/2}$ factor is necessary to make the operator symmetric. It is selfadjoint on all absolutely continuous functions.

For dilations the polar basis is appropriate. The commutation relations and uncertainty principle become:

$$\boxed{[\hat{x}^0, \cos\theta] = [\hat{x}^0, \mathrm{e}^{\mathrm{i}\varphi}] = 0 \ , [x^0, r] = \mathrm{i}\ell r.}$$

$$\boxed{\Delta x^0 \Delta r \geq \frac{\ell}{2}|\langle r\rangle|.}$$

There is a well known (Pauli) reason for which time cannot be an operator in quantum mechanics, since it is conjugate to the Energy $\boxed{[t, H] = \mathrm{i}\hbar}$. Then they must have the same spectrum (Von Neumann), and $\boxed{H}$ has a spectrum bounded from below. While $\boxed{t}$ should have as spectrum the whole real line.

This does not apply here since I am not considering $\boxed{\hbar}$ here. A full theory would of course require $\boxed{\hbar \neq 0}$, and we will have to invent something else.

Look for the spectrum spectrum of the time operator: Monomials in r are formal solutions of the eigenvalue problem:

$$\boxed{\mathrm{i}\ell\left(r\partial_r + \frac{3}{2}\right)r^\alpha = \mathrm{i}\ell(\alpha + \frac{3}{2})r^\alpha = \ell_\alpha r^\alpha,}$$

The eigenvalues are real if and only if $\boxed{\alpha = -\frac{3}{2} + \tau}$ with $-\infty < \tau < \infty$ a real number.

For momentum we had plane waves, in this case we have the following distributions

$$\boxed{T_\tau = \frac{r^{-\frac{3}{2}-\mathrm{i}\tau}}{\ell^{-\mathrm{i}\tau}} = r^{-\frac{3}{2}}\mathrm{e}^{-\mathrm{i}\tau\log\left(\frac{r}{\ell}\right)}}$$

The distribution has the correct dimension of a length $\boxed{3/2}$, the factor of $\boxed{\ell}$ is there to avoid taking the logarithm of a dimensional quantity. Since $\boxed{\ell}$ is a natural scale for the model, its choice is natural, but not unique.

For quantum phase space we had as complete set of observables either three $\boxed{q}$ or three $\boxed{p}$, connected by a Fourier transform,

For $\boxed{\kappa}$-Minkowski we have either $\boxed{(r,\theta,\varphi)}$ or $\boxed{(\tau,\theta,\varphi)}$, and we switch among the two with a Mellin transform

$$\boxed{\psi(r,\theta,\varphi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}\tau\, r^{-\frac{3}{2}} \mathrm{e}^{-\mathrm{i}\tau \log\left(\frac{r}{\ell}\right)} \widetilde{\psi}(\tau,\theta,\varphi) = \mathcal{M}^{-1}\left[\widetilde{\psi}(\tau,\theta,\varphi),\ r\right],}$$

$$\boxed{\widetilde{\psi}(\tau,\theta,\varphi) = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} \mathrm{d}r\, r^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}\tau \log\left(\frac{r}{\ell}\right)} \psi(r,\theta,\varphi) = \mathcal{M}\left[\psi(r,\theta,\varphi), \frac{3}{2} + \mathrm{i}\tau\right].}$$

$\boxed{|\psi|^2}$ and $\boxed{|\widetilde{\psi}|^2}$ can be interpreted as the probabilty density to find the particle in position $\boxed{r}$ or time $\boxed{\tau}$ respectively

It is useful to have an idea of the dimensional quantities involved.

Call $\boxed{t}$ the eigenvalue of the time operator $\boxed{\frac{x^0}{c}}$, then $\boxed{\tau = t\frac{c}{\ell}}$.

$\boxed{\frac{c}{\ell}}$ is a dimensional quantity. If we choose for $\boxed{\ell}$ the Planck length then $\boxed{\frac{c}{\ell} \sim 2 \cdot 10^{43}}$ Hz. In other words if $\boxed{t = 1}$ s, then $\boxed{\tau = 2 \cdot 10^{43}}$, an extremely large number.

If $\boxed{t}$ is of the order of Planck time, then $\boxed{\tau \sim 1}$.

This uncertainty for $\boxed{\kappa}$-Minkowski is different form the previous one, it grows with distance form the origin. Should we worry at the macroscopic level?

The most precise time "click" measurement to date is of the order of $\boxed{10^{-19}}$ sec., assuming that the uncertainty is of this order, for $\boxed{x^0 = ct}$ we have $\boxed{\Delta x^0 \sim 10^{-11}\text{m.}}$

The most precise distance measured (at the Large Hadron Collider) is $\boxed{10^{-18}\text{m.}}$. Assuming it is the uncertainty we have $\boxed{\Delta x^0 \Delta r \sim 10^{-29}\text{m.}^2}$

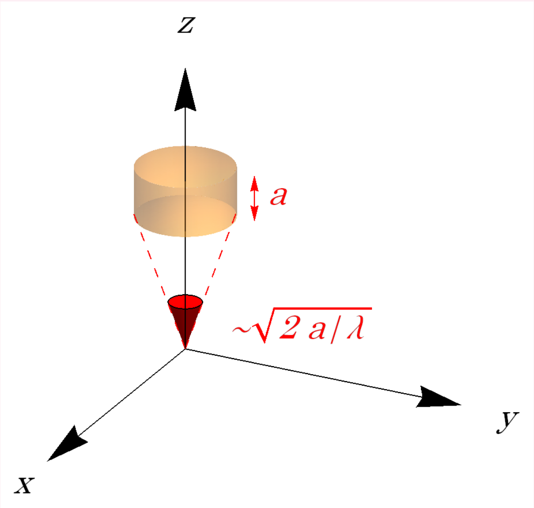
The real killer is the fact that $\boxed{\ell}$ is of the order of Planck Lenght $\boxed{\sim 10^{-35}\text{m.}}$.
I have to start worrying if

$$\boxed{\langle r \rangle \sim 10^6\text{m.}}$$

I will now give some examples of localised state, at the origin and away

Consider the following state (chosen to simplify calculations) localised in space in a small region of size $\boxed{a}$ around a point at distance $\boxed{z_0}$ along the $\boxed{z}$ axis.

$$\boxed{\psi_{z_0,a}(r,\theta,\varphi) = \begin{cases} \sqrt{\frac{3\ell}{2a\pi((a+z_0)^3 - z_0^3)}}, & z_0 \leq r \leq (z_0 + a) \text{ and } \cos\theta > 1 - \frac{a}{\ell} \\ 0, & \text{otherwise} \end{cases}}$$



In the limit $\boxed{a \to 0}$ the state is localised in $\boxed{z_0}$

The Mellin transform of this function, integrating out the angular variables, gives:

$$\boxed{\int |\tilde{\psi}_{z_0,a}|^2 \sin\theta \, d\theta = \left[\frac{a}{4\pi^2 z_0} - \frac{a^2}{8\ell(\pi^2 z_0^2)} + \mathcal{O}(a^3)\right]}$$

This tends to a constant which vanishes as $a \to 0$. Localising in space implies delocalising in time

The series expansion for $\boxed{a}$ around $\boxed{0}$, and $\boxed{z_0}$ around $\boxed{\infty}$, are the same. $\boxed{|\tilde{\psi}_{z_0}|^2 = \frac{\ell}{4\pi^2 z_0} - \frac{a\ell}{8\pi^2 z_0^2} + \frac{a^2\ell(7-4\tau^2)}{192\pi^2 z_0^3} + \mathrm{O}\left(a^3\right)}$

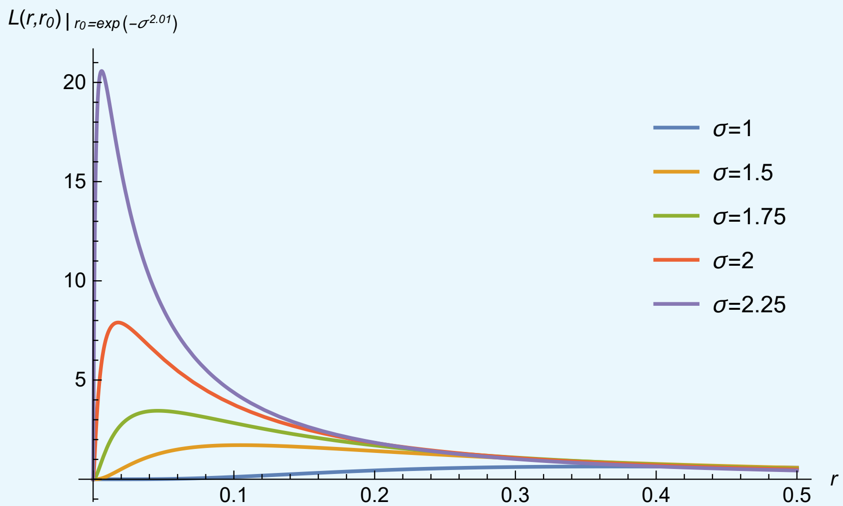
This means that a sharp localization of a particle far away from the origin implies that the particle cannot be localised in time. In accordance with the uncertainty for $\boxed{\kappa}$-Minkowski.

It is impossible to sharply localise a state at a point, except at the origin $\boxed{x^i = 0}$, which is an exceptional point.

The equivalent of the Gaussians of ordinary quantum mechanics are the log-Gaussians

$$\boxed{L(r, r_0) = N e^{-\frac{(\log r - \log r_0)^2}{\sigma^2}} = e^{-\left(\frac{\log\left(\frac{r}{r_0}\right)}{\sigma}\right)^2} \frac{e^{-\frac{9}{16}\sigma^2}}{\sqrt{\sigma}(2\pi)^{3/4}\sqrt{r_0^3}}}$$

They have a maximum in $\boxed{r = r_0}$, which localises at $\boxed{r = r_0}$ as $\boxed{\sigma \to 0}$, and localises at $\boxed{r = 0}$ as $\boxed{r_0 \to 0}$, for any value of $\boxed{\sigma \geq 0}$.



Their Mellin transform are ordinary Gaussians (up to phases and normalizations) independent on $\boxed{r_0}$

$$\boxed{\widetilde{L}(\tau, r_0) = \frac{\sigma^{\frac{1}{2}} e^{-\frac{1}{4}\sigma^2 \tau(\tau - 3i)} r_0^{i\tau}}{2\sqrt[4]{2}\pi^{3/4}}}$$

In the double limit $\boxed{r_0 \to 0}$ and $\boxed{\sigma \to \infty}$, all $\boxed{\langle r^n \rangle_L}$ and all $\boxed{\langle (x^0)^n \rangle_L}$ go to zero as $\boxed{\sigma \to \infty}$.

This is a state localised both in space (at $\boxed{r = 0}$) and in time (at $\boxed{\tau = 0}$)

Localisation at arbitrary time is simply achieved multiplying the state by $\boxed{\left(\frac{r}{\ell}\right)^{\mathrm{i}\tau_0}}$

With the usual abuse of notation we will call these state as $\boxed{|o_\tau\rangle}$.

We have argued that the origin is a special point. Does this mean that somewhere in the universe there is “the origin”. A special position in space singled out by the $\boxed{\kappa}$-God?

Implicitly in our discussion, when we were referring to states we were assuming the existence of an observer measuring the localisation of states.

This observer is located at the origin, and he can measure with absolute precision where she is. For him “here” and “now” make sense. She cannot localise with precision states away from him, as a consequence of the noncommutativity of $\boxed{\kappa}$-Minkowski.

What about other observers? A different observer will be in general Poincaré transformed, i.e. translated, rotated and boosted. These operations are usually performed with an element of the Poincaré group. But now we have $\boxed{\kappa}$-Poincaré!

Require invariance under the transformation $\boxed{x^\mu \to x'^\mu = \Lambda^\mu{}_\nu \otimes x^\nu + a^\mu \otimes 1}$

But now the coordinate functions on the group are noncommutative, they are (in a particular basis, Zakrzewski)

$$\boxed{[a^\mu, a^\nu] = \mathrm{i}\ell\,(\delta^\mu{}_0\, a^\nu - \delta^\nu{}_0\, a^\mu)\ , \qquad [\Lambda^\mu{}_\nu, \Lambda^\rho{}_\sigma] = 0}$$

$$\boxed{[\Lambda^\mu{}_\nu, a^\rho] = \mathrm{i}\ell\left[(\Lambda^\mu{}_\sigma\delta^\sigma{}_0 - \delta^\mu{}_0)\,\Lambda^\rho{}_\nu + \left(\Lambda^\sigma{}_\nu\delta^0{}_\sigma - \delta^0{}_\nu\right)\eta^{\mu\rho}\right].}$$

In particular notice that translations are now noncommuting. With the same commutation relations of the coordinates.

We represented the $\boxed{\kappa}$-Minkowski algebra as operators. But in doing so we had implicitly chosen an observer.

In order to take into account the fact that there are different observers we enlarge the algebra (and consequently the space) to include the parameters of the new observers. We call then new set of states as $\boxed{\mathcal{P}_\kappa}$

Our (generalised) Hilbert space will now comprise not only functions on space-time (either functions of $\boxed{r}$ or $\boxed{\tau}$), but also functions of the $\boxed{a}$'s and $\boxed{\Lambda}$'s.

We can represent the $\boxed{\kappa}$-Poincaré group faithfully as

$$\boxed{a^\rho = -\mathrm{i}\,\frac{\ell}{2}\left[\left(\Lambda^\mu{}_\sigma \delta^\sigma{}_0 - \delta^\mu{}_0\right)\Lambda^\rho{}_\nu + \left(\Lambda^\sigma{}_\nu \delta^0{}_\sigma - \delta^0{}_\nu\right)\eta^{\mu\rho}\right]\Lambda^\nu{}_\alpha \frac{\partial}{\partial \omega^\mu{}_\alpha} + \mathrm{i}\frac{\ell}{2}\left(\delta^\rho{}_0\, q^i \frac{\partial}{\partial q^i} + \delta^\mu{}_i\, q^i\right) + \frac{1}{2}\mathrm{h.c.}}$$

Where $\boxed{\omega}$ are the parameters of the Lorentz transformation, and the $\boxed{\Lambda}$'s are represented as multiplicative operators

We have therefore that, like spacetime, the space of observers is also noncommutative, and the noncommutativity is only present in the translation sector.

We now explore the space of observers, seen as states. First consider the observer located at the origin, which is reached via the identity transformation.

Define $\boxed{|o\rangle_{\mathcal{P}}}$ with the property:

$$\boxed{{}_{\mathcal{P}}\langle o|\, f(a,\Lambda)|o\rangle_{\mathcal{P}} = \varepsilon(f)\,,}$$

with $\boxed{f(a,\Lambda)}$ a generic noncommutative function of translations and Lorentz transformation matrices, and $\boxed{\varepsilon}$ the counit.

This state describes the Poincaré transformation between two coincident observers. The state is such that all combined uncertainties vanish. Coincident observers are therefore a well-defined concept in $\boxed{\kappa}$-Minkowski spacetime.

A change of observer will transform $\boxed{x^\mu \to x'^\mu = \Lambda^\mu{}_\nu \otimes x^\nu + a^\mu \otimes 1}$ and primed and unprimed coordinates correspond to different observers.

Identifying $\boxed{x}$ with $\boxed{\mathbb{1} \otimes x}$ we generate an extended algebra $\boxed{\mathcal{P} \otimes \mathcal{M}}$ which extends $\boxed{\kappa}$-Minkowski by the $\boxed{\kappa}$-poincaré group algebra.

This algebra takes into account position states and observables

Remember that, just as we cannot sharply localise position states, neither we can sharply localise where the observer is.

Since Lorentz transformations commute among themselves, we can however say if two observers are just rotated with respect to each other

We can build the action of the position, translation and Lorentz transformations operator on generic functions of all those variables.

To simplify notations let us consider $\boxed{1+1}$ dimensions. In this case there are only two position coordinates, two translations coordinates and one Lorentz transformation parametrized by $\boxed{\xi}$

The relations are $\boxed{\Lambda^0{}_0 = \Lambda^1{}_1 = \cosh\xi\,,\ \ \Lambda^0{}_1 = \Lambda^1{}_0 = \sinh\xi\,,}$

$$\boxed{[a^0, a^1] = \mathrm{i}\ell\, a^1\,, \qquad [\xi, a^0] = -\mathrm{i}\ell \sinh\xi\,, \qquad [\xi, a^1] = \mathrm{i}\ell\,(1-\cosh\xi)\ .}$$

And the action on $\boxed{\mathcal{P}}$ is

$$\boxed{a^0 = \mathrm{i}\ell q \frac{\partial}{\partial q} + \mathrm{i}\ell \sinh\xi\, \frac{\partial}{\partial \xi}\,,\ \ a^1 = q + \mathrm{i}\ell\,(\cosh\xi - 1)\ \frac{\partial}{\partial\xi}\,,}$$

States (non entangled) will be objects of the kind $\boxed{|g\rangle \otimes |f\rangle}$

In particular $\boxed{|g\rangle \otimes |o\rangle}$ is a pure translation of the state at the origin.

The new observer measures coordinates with $\boxed{x'}$. The expectation values on (normalised) transformed state is

$$\boxed{\langle x'^{\mu}\rangle = \langle g| \otimes \langle o|x'^{\mu}|g\rangle \otimes |o\rangle = \langle g|\Lambda^{\mu}{}_{\nu}|g\rangle\langle o|x^{\nu}|o\rangle + \langle g|a^{\mu}|g\rangle\langle o|o\rangle\,,}$$

We get:

$$\boxed{\langle x'^{\mu}\rangle = \langle g|a^{\mu}|g\rangle\,,}$$

The expectation value of the transformed coordinates is completely defined by translations. This is natural, the different observers are comparing positions, not directions.

In general

$$\boxed{\langle x'^{\mu_1} \ldots x'^{\mu_n} \rangle = \langle g | a^{\mu_1} \ldots a^{\mu_n} | g \rangle \langle o | o \rangle = \langle g | a^{\mu_1} \ldots a^{\mu_n} | g \rangle \,.}$$

Poincaré transforming the origin state $\boxed{|o\rangle}$ by a state with wave function $\boxed{|g\rangle}$ in the representation of the $\boxed{\kappa}$-Poincaré algebra, the resulting state will assign, to all polynomials in the transformed coordinates the same expectation value as what assigned by $\boxed{|g\rangle}$ to the corresponding polynomials in a^μ.

In other words, the state $\boxed{x'^\mu}$ is identical to the state of $\boxed{a^\mu}$.

All uncertainty in the transformed coodinates $\boxed{\Delta x'^\mu}$ is introduced by the uncertainty in the state of the translation operator, $\boxed{\Delta a^\mu}$.

It is also possible to see that the uncertainty of states increases with translation.

I can summarise saying that all observers can sharply localise states in their vicinity, and cannot localise states far away from them.

The apparent paradox of a state badly localisable by Alice, but which is well localised by Bob, is that Bob herself is badly localised by Alice, and of course viceversa.

All this is qualitatively perfectly compatible with the principle of relative locality (Amelino-Camelia, Kowalski-Glickman, Freidel, Smolin), which however starts in a quite different context: curved momentum space. In this analysis instead momentum does not appear explicitly, although it is present in the symmetry.

One of the tenets of Quantum Mechanics is that the observer is classical, usually macroscopic, and that therefore we “know” how to deal with them.

In quantum gravity this may not be the case. While it is true that the smallness of the Planckian constants suggests this, there may be amplifying effects, and conceptual aspects to deals with.

The group algebra approach, where the parameters of the Poincaré transformations do not commute is the key to understand the observer-dependent transformations

Transformations relating different frames belong to a noncommutative algebra. Hence localisability limitations.

$\boxed{\varrho}$-Minkowski

This time the uncertainty will be between time and the angular variable. And one should definitely resist the temptation to write:

$$\boxed{\cancel{\Delta t \Delta \varphi \geq \frac{\varrho}{2}}}$$

In the $\boxed{\{\rho, z, \varphi\}}$ basis $\boxed{t}$ is represented by the derivation operator $\boxed{-\mathrm{i}\varrho\partial_\varphi}$.

This operator has Discrete Spectrum!

A change of basis is given by the Fourier series. The eigenstates of momentum are $\boxed{\mathrm{e}^{\mathrm{i}n\varphi}}$, and they are completely delocalised in $\boxed{\varphi}$

On the other hand, a state completely localised in $\boxed{\varphi}$, given by a $\boxed{\delta}$, which requires a superposition with equal weights of all eivenvalues of time.

$$\boxed{\delta(\varphi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}n\varphi}}$$

After a time measurement, which has given as result $\boxed{n_0\varrho}$, the system is in the eigenstate $\boxed{\mathrm{e}^{\mathrm{i}n_0\varphi}}$.

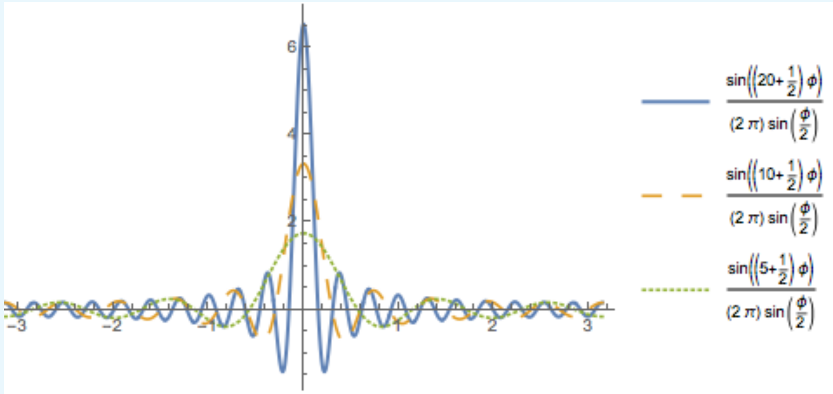
A slightly uncertain state uses a great number of Fourier modes to built a state peaked around some time, then the corresponding uncertainty is the angular variable is given by the fact that only a finite set of elements of the basis are available.

For $\boxed{\varrho}$ Planckian of the quantum of time (also called a chronon), is $\boxed{5.39\ 10^{-44}\, sec.}$

The most accurate measurement of time is $\boxed{\sim 10^{-19}\, sec.}$ Heuristically the superposition of $\boxed{10^{35}}$ quanta of time is needed.

Approximate $\boxed{\delta}$ by the Dirichlet nucleus $\boxed{\delta_N = \sum_{n=-N}^{N} \mathrm{e}^{in\varphi} = \frac{1}{2\pi} \frac{\sin(N+\frac{1}{2})\varphi}{\sin \frac{N}{2}\varphi}}$

For $\boxed{N = 5, 10, 15}$.



The needs $\boxed{N \sim 10^{35}}$. Then the first zero of the nucleus is at $\boxed{\varphi \sim 10^{-35}}$. We may assume this to be the uncertainty in an angle determination. To translate this as an uncertainty in position we need $\boxed{\rho}$. For the radius observable universe ($\boxed{10^{26}\, m}$) the uncertainty is of the order of one metre.

Is this all pervading clicking a feature of our universe? Is time translation definitely lost? Putting time on a lattice may be disturbing.

Self-adjointness come to the rescue. Anybody who has studied the Aharonov-Bohm experiment knows that the momentum operator on a compact domain is a rich operator.

It is self-adjoint on periodic functions, but is also selfadjoint on functions periodic up to a phase. In this case the eigenfunctions are $\boxed{\mathrm{e}^{\mathrm{i}(n+\alpha)\varphi}}$.

The differences between states is unchanged, and the effect is a rigid shift. This however means that a different choices of selfadjointess domains. Time translations are undeformed, and two time translated observers will be in different, but equivalent domains.

Some comments now on work in progres for $\boxed{\theta}$-Poincaré. This is the space-time of the well-known Grönewold-Moyal product, of the Doplicher-Fredenhagen-Roberts space. For it the commutation is

$$\boxed{[x^\mu, x^\nu] = \mathrm{i}\theta^{\mu\nu}}$$

The constant $\boxed{\theta}$ is considered a fundamental quantity, with the dimension of a length squared and being antisymmetric gives rise to fundamental vector and pseudovectors ("electric" and "magnetic"). They are fixed and as before we need to use a quantum Hopf algebra.

This is usually obtained via a Drinfeld twist. I will not go into details of this.

Such particular directions should show up in the cosmic background as a quadrupole effect, and present data are pushing its scale above to the Planck scale.

There is one fundamental difference between the quantum group in this case and the two discussed so far.

In this case the commutation relations among the translations parameters do not reproduce the commutation relations among coordinates.

In other words it is impossible to obtain spacetime as a quotient.

The relations are in fact complicated:

$$\boxed{[a^\mu, a^\nu] = -\mathrm{i}\alpha^2\theta^{\rho\sigma}(\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma - \delta^\mu{}_\rho\delta^\nu{}_\sigma)}$$

with the Lorentz parameters $\boxed{\Lambda^\mu{}_\nu}$ central elements.

The Hilbert space this time will be made of functions of two of the $\boxed{x}$'2. Say $\boxed{x^0, x^1}$, the $\boxed{\Lambda}$'s, which are diagonal, and two linear combinations of the $\boxed{a}$'s, the ones which put the commutation relations in normal form.

Imagine now that two observers make an experiment (or an observation) which find $\boxed{\theta}$.

In order to compare their findings they will necessarily have to use a translation operator to compare the result. They will also have to make sure that are using the same reference frame, to express the components.

The translation operator will be something like $\boxed{\mathrm{e}^{\mathrm{i}a^\mu \partial_{x^\mu}}}$. And will act on vector of the Hilbert space.

The problem is that the simple translation operator acts in an elaborated way, inducing rotations and boosts, so that the comparison becomes quite complicated.

I am not sure in the end the comparison can be made ina meaningful way. Stay tuned for more results!

Quantum observers have appeared recently in the literature also form the information theory point of view, for example in the work of Brukner, Giacomini, Castro-Ruiz, Höhn. . .

There the emphasis is in another fundamental quantum aspect: entanglement.

Not only a state can be in an entangled state, but also the reference frame can be entangled. It becomes necessary therefore to define transformations between frames which are fully quantum objects.

Localisability becomes therefore a frame dependent concept, and this leads to a quantum spacetime.

This approach is therefore in some sense dual to the one I proposed here.

Final Remarks

The main message I want to convey is that quantum gravity will require Quantum Spacetime.

Quantum Spacetime in turn requires quantum observers.

This is of course true for quantum phase space as well. There we became (more or less) used to deal with the contradictions of the quantum/classical interaction. We learned how to deal with noncommuting observables for example.

But a quantum spacetime will pose further challenges and other layers to our understanding.

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