Matrix phase in non-commutative space: Between theory and numerical experiments

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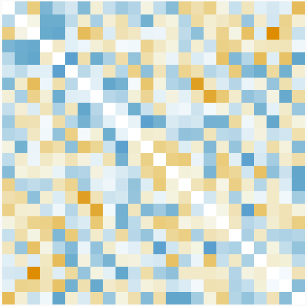


Quantum & Fuzzy*, Belgrade, 04/04/2024

*Workshop in honour of the 65th birthday of Professor Maja Burić

Timeline

- gauge model on the truncated Heisenberg algebra:
  - Faculty of Physics, Belgrade, ≤ ’16 — Burić, Nenadović

- phase transitions in matrix models on NC space:
  - DIAS, ’17, ’18 — COST STSM MP1405 — O’ Connor, Kováčik
  - Faculty of Pharmacy, Belgrade, ’19, ’22 — Vasović, Ranković
  - Comenius University in Bratislava, ≥’21 — Tekel, Kováčik

- based on arXiv 2002.05704 + some new stuff
  2104.00657
  2209.00592

Outline

1. GW model
2. Matrix model & Phase transitions
3. Analytical results
4. Conclusions & Outlook

GW model

$$S_{GW} = \int d^{2n}x \left(\frac{1}{2}\partial_\mu \phi \star \partial^\mu \phi + \frac{m^2}{2}\phi \star \phi + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi + \right.$$
$$\left. + \frac{1}{2}\Omega^2 (\tilde{x}_\mu \phi) \star (\tilde{x}^\mu \phi) \right)$$

where $[x^\mu, x^\nu]_\star = i\theta^{\mu\nu}$ and $\tilde{x}_\mu = 2\left(\theta^{-1}\right)_{\mu\nu} x^\nu$

- superrenormalizable in 2-dim (Grosse and Wulkenhaar 03)
- renormalizability of the 2-dim $\lambda\phi^4_\star$-model is restored by defining it as a $\Omega \to 0$ limit of the series of GW models in which Ω itself does not renormalize and serves as a series-label
- harm. osc. potential $\Longleftrightarrow$ curvature (Burić and Wohlgenannt 10)

Heisenberg algebra

- absorbing the length scale, going to matrices:

$$[x^\mu, x^\nu]_\star = i\theta^{\mu\nu} \qquad \Longrightarrow \qquad [X, Y] = i\,\mathbb{1}$$

- infinite-dimensional representation, $\quad X \to + \quad Y/i \to -$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} & \pm 1 & & & \\ +1 & & \pm\sqrt{2} & & \\ & +\sqrt{2} & & \pm\sqrt{3} & \\ & & +\sqrt{3} & & \cdots \\ & & & \vdots & \end{bmatrix}$$

$$[X, Y] = i\,(\mathbb{1} - Z) \qquad [X, Z] = i\,\{Y, Z\} \qquad [Y, Z] = -i\,\{X, Z\}$$

- possible to define diff. calculus (Burić and Wohlgenannt 10)
- derivatives given by NC coordinates:

$$\partial_\mu = [iP_\mu, \cdot\,] \qquad P_1 = -Y \qquad P_2 = X \qquad P_3 = \frac{\mathbb{1}}{2} - Z$$

- curvature:

$$R = \frac{15}{2}\mathbb{1} - 2Z - 4\left(X^2 + Y^2\right) \approx -16\,\mathrm{diag}(1, 2, \ldots, N)$$

Matrix model & Phase transitions

- Weyl transform of the GW model:

$$\phi \longleftrightarrow \Phi \qquad \int \longleftrightarrow \sqrt{\det 2\pi\theta}\ \mathrm{tr}$$

- matrix model on the truncated Heisenberg algebra:

$$\boxed{S_N = N\,\mathrm{tr}\Big(\Phi\widehat{K}\Phi - g_r R\Phi^2 - g_2\Phi^2 + g_4\Phi^4\Big)}$$

- kinetic operator: $\widehat{K} = [X,[X,\cdot\,]] + [Y,[Y,\cdot\,]]$
- unscaled vs. scaled parameters:

$$G_2 = Ng_2 \qquad G_4 = Ng_4$$

$$X \to \frac{X}{\sqrt{N}} \qquad Y \to \frac{Y}{\sqrt{N}} \qquad R \to \frac{R}{N}$$

- EOM:

$$2\widehat{K}\Phi - g_r\left(R\Phi + \Phi R\right) + 2\Phi\left(2g_4\Phi^2 - g_2\,\mathbb{1}\right) = \mathbb{0}$$

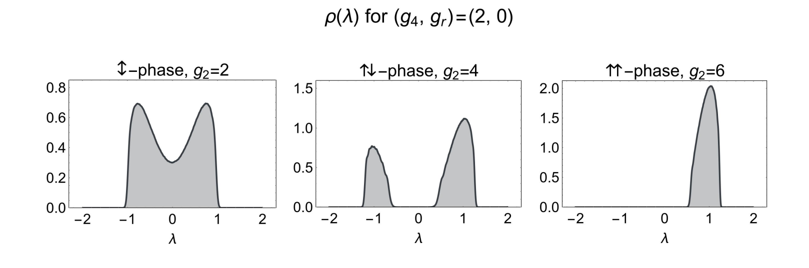
- vacuum solutions:

$$\Phi = \frac{\operatorname{tr}\Phi}{N}\,\mathbb{1} \qquad \Phi = \mathbb{0} \qquad \Phi^2 = \begin{cases} \mathbb{0} & \text{for } g_2 \leq 0 \\ \frac{g_2\mathbb{1}}{2g_4} & \text{for } g_2 > 0 \end{cases}$$

- 3 phases:
  - disordered/1-cut symmetric phase: $\Phi_{\updownarrow} = \mathbb{0}$
  - **striped/2-cut phase:** $\Phi_{\uparrow\downarrow} \propto U\,\mathbb{1}_{\pm}\,U^{\dagger}$
  - ordered/1-cut asymmetric phase: $\Phi_{\uparrow\uparrow} \propto \mathbb{1}$

- **modified ordered phases:**

$$\Phi_R^2 = \frac{g_2\mathbb{1} + g_r R}{2g_4}$$

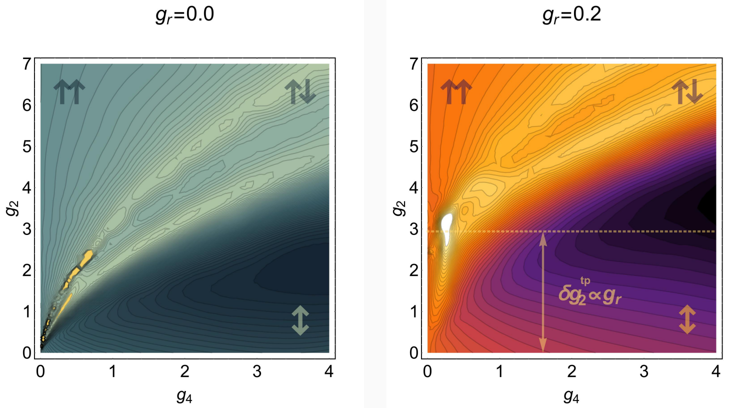


- disordered/1-cut symmetric phase: $\Phi_{\updownarrow} = \mathbb{0}$
- striped/2-cut phase: $\Phi_{\uparrow\downarrow} \propto U \, \mathbb{1}_{\pm} \, U^{\dagger}$
- ordered/1-cut asymmetric phase: $\Phi_{\uparrow\uparrow} \propto \mathbb{1}$

- we are interested in:

$$\langle O \rangle = \frac{\int [d\Phi]\, O\, e^{-S_N}}{\int [d\Phi]\, e^{-S_N}} \qquad \operatorname{Var} O = \langle O^2 \rangle - \langle O \rangle^2$$

- numerical integration by Hamiltonian Monte Carlo, $N \leq 70$
- thermodynamic quantities:
  - heat capacity $C = \operatorname{Var} S/N^2$
  - susceptibility $\chi = \operatorname{Var} \langle |\operatorname{tr} \Phi| \rangle / N^2$
- random field characteristics:
  - eigenvalue (ρ_λ) and trace (ρ_{tr}) probability distributions



$$S_N \ni \mathrm{tr}\left(g_r|R|\Phi^2\right) \Rightarrow \delta g_2^{\mathrm{tp}} \leq 16 g_r \qquad \delta g_2^{\mathrm{tp}} \geq 16 g_r \Leftarrow \Phi_R^2 = \frac{g_2\mathbb{1} - g_r|R|}{2g_4}$$

$$\boxed{\delta g_2^{\mathrm{tp}} = 16 g_r}$$

- triple-point shift:

$$\delta g_2^{\text{tp}} \propto g_r \quad \Longrightarrow \quad \delta G_2^{\text{tp}} \propto N g_r \quad \Longrightarrow \quad \boxed{G_2^{\text{tp}} \sim N \to \infty}$$

- mass shift (δm^2_{ren}: e.g. Franchino-Viñas and Pisani 14):

$$\delta m^2_{\text{ren}} = \frac{\lambda}{12\pi(1+\Omega^2)} \log \frac{\Lambda^2\theta}{\Omega} \quad \Longrightarrow \quad \boxed{\left|\delta G_2^{\text{ren}}\right| \sim \log N < G_2^{\text{tp}}}$$

- $\lambda\phi^4_\star$-renormalization (Grosse and Wulkenhaar 03):

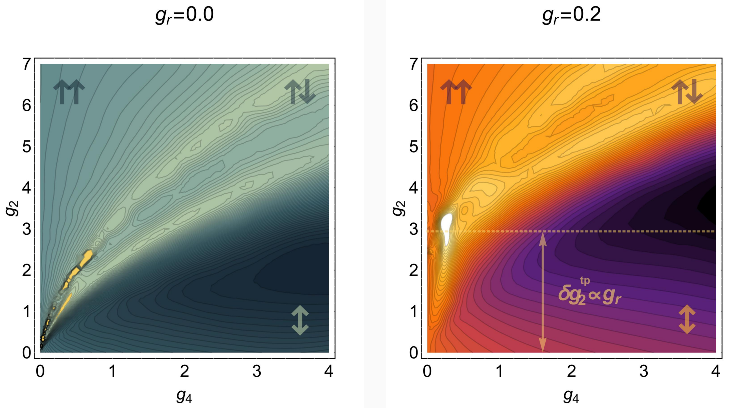
$$\Lambda^2 \sim N, \quad g_r \sim \frac{1}{\log^2 N} \to 0 \quad \Rightarrow \quad \text{GW} \to \lambda\phi^4_\star$$

- triple-point shift:

$$\delta g_2^{\text{tp}} \propto g_r \quad \Rightarrow \quad \delta G_2^{\text{tp}} \propto N g_r \quad \Rightarrow \quad \boxed{G_2^{\text{tp}} \sim \frac{N}{\log^2 N} \to \infty}$$

- mass shift:

$$\delta m^2_{\text{ren}} = \frac{\lambda}{12\pi(1+\Omega^2)} \log \frac{\Lambda^2\theta}{\Omega} \quad \Rightarrow \quad \boxed{\left|\delta G_2^{\text{ren}}\right| \sim \log N < G_2^{\text{tp}}}$$



$$\left|\delta G_2^{\text{ren}}\right| < G_2^{\text{tp}} \quad \Longrightarrow$$ bare model stays in ↕-phase!

Analytical results

$$S_N(\Phi) = N \ \mathrm{tr}\Big(\Phi \widehat{K} \Phi - g_r R \Phi^2 - g_2 \Phi^2 + g_4 \Phi^4\Big)$$

$$S_N(\Lambda, U) = N \ \mathrm{tr}\Big((U \Lambda U^\dagger) \widehat{K} (U \Lambda U^\dagger) - g_r R U \Lambda^2 U^\dagger - g_2 \Lambda^2 + g_4 \Lambda^4\Big)$$

$$S_{\text{eff}}(\Lambda) = N \ \mathrm{tr}\Big(\textbf{?} - g_2 \Lambda^2 + g_4 \Lambda^4\Big) - \sum_{m \neq n} \log |\lambda_m - \lambda_n|$$

$$\delta S_{\text{eff}} = 0 \quad \Rightarrow \quad \textbf{?} - g_2 \lambda + 2 g_4 \lambda^3 = \int_{\text{support}} d\lambda' \, \frac{\rho(\lambda')}{\lambda - \lambda'}$$

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example: $\rho(\lambda) = \dfrac{g_4(r^2 + 2\lambda^2) - g_2}{\pi} \sqrt{r^2 - \lambda^2}$

$$Z = \int [d\Phi]\, e^{-S_N} = \int [d\Lambda]\, \Delta^2(\Lambda)\, e^{-N\,\mathrm{tr}\left(-g_2\Lambda^2 + g_4\Lambda^4\right)} \int [dU]\, e^{g_r N\,\mathrm{tr}\left(URU^\dagger\Lambda^2\right)}$$

$$I = \int_{U(N)} [dU]\, e^{t\,\mathrm{tr}\left(AUBU^\dagger\right)} = \frac{c_N}{t^{N(N-1)/2}\Delta(A)\Delta(B)} \begin{vmatrix} e^{ta_1b_1} & e^{ta_1b_2} & \cdots & e^{ta_1b_N} \\ e^{ta_2b_1} & e^{ta_2b_2} & \cdots & e^{ta_2b_N} \\ \vdots & \vdots & & \vdots \\ e^{ta_Nb_1} & e^{ta_Nb_2} & \cdots & e^{ta_Nb_N} \end{vmatrix}$$

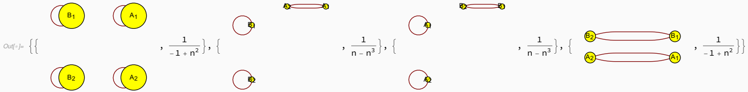
$$c_N = \prod_{k=1}^{N-1} k! \qquad \Delta(A) = \prod_{1\le i<j\le N} (a_j - a_i) \qquad a, b - \text{eigenvalues}$$

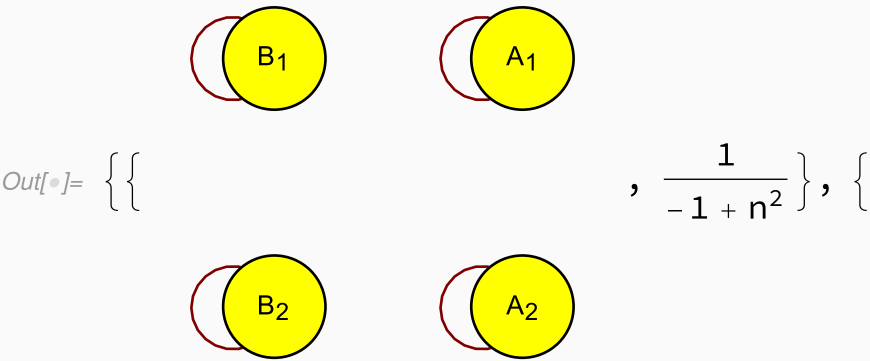
$$I = \sum_{n=0}^{\infty} \frac{t^n}{n!} I_n \qquad I_n = \int [dU] \, \mathrm{tr}^n\left(AUBU^\dagger\right)$$

$$I = \exp\left(-\sum_{n=1}^{\infty} \frac{t^n}{n!} S_n\right)$$

$$S_1 = -I_1 \quad \Rightarrow \quad S_2 = S_1^2 - I_2 \quad \Rightarrow \quad S_3 = -S_1^3 - 3S_1S_2 - I_3 \quad \Rightarrow \quad \ldots$$

- Fukuda M, König R, Nechita I. RTNI—A symbolic integrator for Haar-random tensor networks. J. Phys. A: Math. Theor. 52 (2019)
- *Mathematica* or *Python*, up to 20 U-matrices


Out[•]= {{
B1
A1
B2
A2
, 1/(-1 + n^2)}, {
, 1/(n - n^3)}, {
, 1/(n - n^3)}, {
B2
B1
A2
A1
, 1/(-1 + n^2)}}


Out[•]= {{
B1
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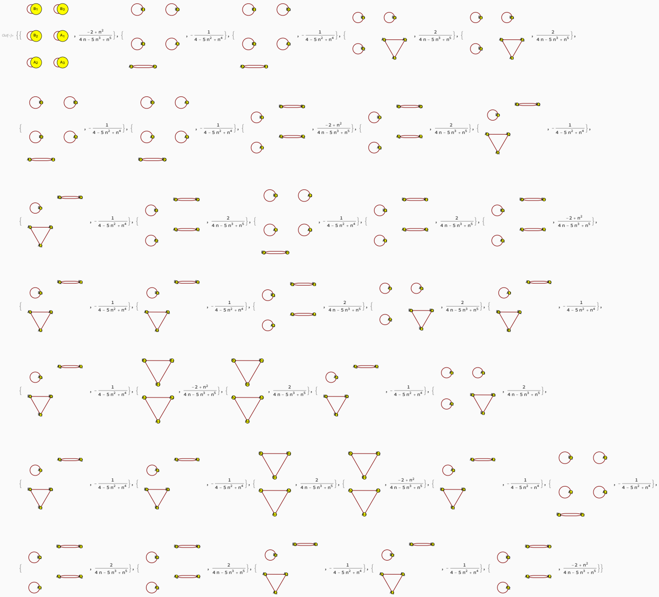
, 1/(-1 + n^2)}, {



	I_1	I_2	I_3	I_4	I_5	I_6
terms	1	4	36	576	14,400	518,400
time*	0.2 s	0.3 s	1 s	19 s	10 min	21 h

*Ryzen 7, 16 GB RAM + 20 GB swap, Ubuntu 20, Python

- additional Python code for collecting the repeating terms
- in the end, I_n has p_n^2 terms: 1, 4, 9, 25, 49, 121, ...

Effective action corrections

$$S_1 = -\frac{1}{N} \cdot \mathrm{tr}A \cdot \mathrm{tr}B$$

$$S_2 = -\frac{1}{N^2 - 1^2} \cdot \mathrm{tr}(A - \mathrm{tr}A/N)^2 \cdot \mathrm{tr}(B - \mathrm{tr}B/N)^2$$

$$S_3 = -\frac{2N}{(N^2 - 1^2)(N^2 - 2^2)} \cdot \mathrm{tr}(A - \mathrm{tr}A/N)^3 \cdot \mathrm{tr}(B - \mathrm{tr}B/N)^3$$

$$A_n = \frac{\mathrm{tr}(A - \mathrm{tr}A/N)^n}{N} \qquad B_n = \frac{\mathrm{tr}(B - \mathrm{tr}B/N)^n}{N}$$

$$S_4 = -\frac{6N^2(N^2+1)}{(N^2-1^2)(N^2-2^2)(N^2-3^2)} \cdot A_4\ B_4$$
$$-\frac{18N^2(N^2+1)(N^2-3)}{(N^2-1^2)^2(N^2-2^2)(N^2-3^2)} \cdot A_2^2\ B_2^2$$
$$+\frac{6N(2N^2-3)}{(N^2-1^2)(N^2-2^2)(N^2-3^2)} \cdot \left(A_4\ B_2^2 + B_4\ A_2^2\right)$$

$$S_5 = -\frac{24N\left(N^2+5\right)}{(N^2-1^2)(N^2-2^2)(N^2-3^2)(N^2-4^2)} \cdot A_5\ B_5$$
$$-\frac{480\left(N^4-5N^2-1\right)}{N(N^2-1^2)^2(N^2-2^2)(N^2-3^2)(N^2-4^2)} \cdot A_3\ A_2\ B_3\ B_2$$
$$+\frac{120\left(N^2-2\right)}{(N^2-1^2)(N^2-2^2)(N^2-3^2)(N^2-4^2)} \cdot (A_5\ B_3\ B_2 + B_5\ A_3\ A_2)$$

- for diagonal A, transpose w.r.t. anti-diagonal is:

$$A_T = \sigma_1 A \sigma_1$$

- for $A = -A_T$ (this is the case for $R - \mathrm{tr}R/N$):

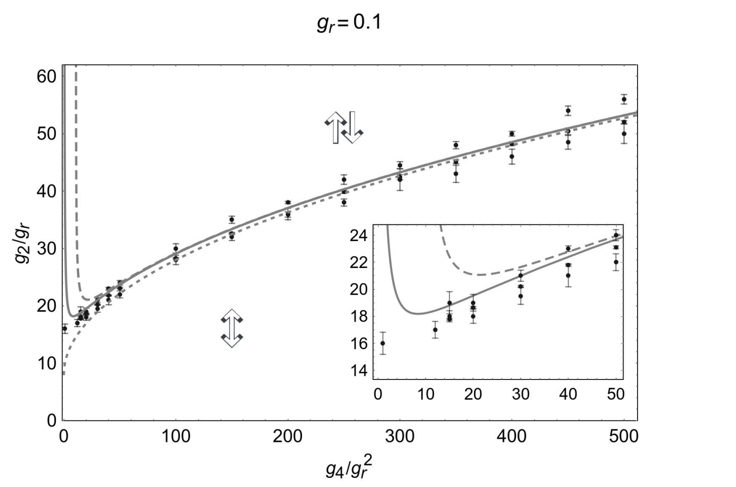
$$\begin{aligned}\int [dU]\,\mathbf{tr}^n\left(UAU^\dagger B\right) &= \int [d(U\sigma_1)]\,\mathbf{tr}^n\left[(U\sigma_1)A(U\sigma_1)^\dagger B\right] \\ &= \int [d(U\sigma_1)]\,\mathbf{tr}^n\left[U(\sigma_1 A\sigma_1)U^\dagger B\right] \\ &= \int [dU]\,\mathbf{tr}^n\left(UA_T U^\dagger B\right) = (-1)^n \int [dU]\,\mathbf{tr}^n\left(UAU^\dagger B\right)\end{aligned}$$

- when n is odd:

$$\int [dU]\,\mathbf{tr}^{\mathrm{odd}}\left(UAU^\dagger B\right) = 0$$

- S_{odd} couples odd with even orders & $S_1 = -I_1$, hence: $\boxed{S_{\mathrm{odd}} = 0}$

$$
\begin{aligned}
S_{\text{eff}}(\Lambda) = &- (g_2 - 8g_r)\, N\, \mathrm{tr}\Lambda^2 \\
&+ \left(g_4 - \frac{32}{3} g_r^2\right) N\, \mathrm{tr}\Lambda^4 + \frac{32}{3} g_r^2\, \mathrm{tr}^2\Lambda^2 \\
&+ \frac{1024}{45} g_r^4 N\, \mathrm{tr}\Lambda^8 + \frac{1024}{15} g_r^4\, \mathrm{tr}^2\Lambda^4 - \frac{4096}{45} g_r^4\, \mathrm{tr}\Lambda^6\, \mathrm{tr}\Lambda^2 \\
&+ \frac{524288}{945} g_r^6\, \mathrm{tr}\Lambda^2\, \mathrm{tr}\Lambda^{10} - \frac{262144}{189} g_r^6\, \mathrm{tr}\Lambda^4\, \mathrm{tr}\Lambda^8 + \frac{524288}{567} g_r^6\, \mathrm{tr}^2\Lambda^6 \\
&- \log \Delta^2(\Lambda)
\end{aligned}
$$



$$g_2 = 2\sqrt{g_4} + 8g_r + \frac{32}{3}\frac{g_r^2}{\sqrt{g_4}} + \frac{256}{15}\frac{g_r^4}{g_4\sqrt{g_4}} - \boxed{\frac{4096}{21}\frac{g_r^6}{g_4^2\sqrt{g_4}}}$$

Conclusions & Outlook

space	model	mass shift	triple point	start. phase	renorm.
$\mathbb{R}^2_\theta$, tHA	$\lambda\phi^4_\star$	UV/IR	0	$\uparrow\downarrow$	NO
$\mathbb{R}^2_\theta$, tHA	GW	$\log N$	N	$\updownarrow$	YES
$\mathbb{R}^2_\theta$, tHA	$\lambda\phi^4_{\mathrm{GW}}$	$\log N$	$N/\log^2 N$	$\updownarrow$	YES
ϵ-tHA	$U(1)$		one phase?	$\uparrow\downarrow$?	NO

- hypothesis: **no $\uparrow\downarrow$-phase $\Longleftrightarrow$ renormalizability**
- connection is **demonstrated** between the **GW model** renormalizability and its phase structure

- analytical front:
  - kinetic-term effects
  - $\uparrow\downarrow \to \uparrow\uparrow$ transition line
- numerical front:
  - gauge model, Chern-Simons
  - 4-dim GW
- metodology front:
  - method extendable to other existing models
  - GPU utilization (CUDA)
  - ML utilization
- speculations: astrophysics, superfluid dark matter

- numerical simulation of phase transitions could tell us in advance about the renormalization properties of new models

The End

Thank you for your attention.

Appendix

Gauge model

- arXiv: 1003.2284, 1203.3016, 1610.01429
- start with the 3-dim action on ϵ-tHA:

$$S_{YM} = -\frac{1}{16g^2}\,\mathrm{tr}(F(*F) + (*F)F)$$

- "compactification" to $z = 0$:

$$S_{YM} = \frac{1}{2}\mathrm{tr}\Big((1-\epsilon^2)(F_{12})^2 + (D_1\phi)^2 + (D_2\phi)^2 + (5-\epsilon^2)\mu^2\phi^2$$
$$- 2(1-\epsilon^2)\mu F_{12}\phi - 4\epsilon F_{12}\phi^2 - \epsilon^2\{p_1 + igA_1, \phi\}^2 - \epsilon^2\{p_2 + igA_2, \phi\}^2\Big)$$

$$D_\alpha\phi = [p_\alpha, \phi] + ig[A_\alpha, \phi] \qquad F_{12} = [p_1, A_2] - [p_2, A_1] + ig[A_1, A_2]$$

- EOM:

$$-(1-\epsilon^2)\mu F_{12} + (5-\epsilon^2)\mu^2\phi + 2i\epsilon g\{F_{12},\phi\} - D^\alpha D_\alpha\phi - \epsilon^2\{X^\alpha,\{X_\alpha,\phi\}\} = 0$$

$$\frac{(1-\epsilon^2)\epsilon^{\alpha\beta}}{g}D_\beta(F_{12}-\mu\phi) + 2i\epsilon\epsilon^{\alpha\beta}\{D_\beta\phi,\phi\} - \left[D^\alpha\phi,\phi\right] - \epsilon^2\{\{X^\alpha,\phi\},\phi\} = 0$$

$$\text{where} \quad X_\alpha = p_\alpha + igA_\alpha$$

- 2 classical vacua, both with $S = 0$:

$$A_1 = 0 \qquad A_2 = 0 \qquad \phi = 0$$

$$A_1 = -\frac{\mu^2 y}{\epsilon g} \qquad A_2 = +\frac{\mu^2 x}{\epsilon g} \qquad \phi = \frac{\mu}{\epsilon g}$$

- ↑↓-phase?

- $\phi\phi$-sector:

$$\epsilon^2\pi^3\mu^2g^2\Lambda^2 \int \phi \,\Box^{-1}\, \phi$$

$$(8/\epsilon^2 - 14 + \epsilon^2)\pi^3\mu^4g^2 \log\Lambda \int \phi \,\Box^{-2}\, \phi$$

- AA-sector:

$$-\frac{(1-\epsilon^2)\pi^3\mu^2g^2}{\epsilon}\Lambda \log\Lambda \int A^\mu \,\Box^{-1}\, A_\mu$$

- nonrenormalizable

- Chern-Simons term:

$$S_{CS} = \frac{\alpha\mu}{3}\,\mathrm{tr}\left(-i(3-\epsilon^2)\left(F_{12} - \frac{\mu^2}{\epsilon}\right) + \frac{2\epsilon}{3}X^\alpha X_\alpha\left(\phi - \frac{\mu}{2\epsilon}\right)\right)$$

- no trivial vacuum
- striped vacuum:

$$S_{CS} = 0 \qquad\qquad S_{YM} + S_{CS} = 0$$

- translationally invariant nontrivial vacuum @ $\epsilon = 1$, $\alpha = 6$:

$$A_1 = A_2 = 0 \qquad\qquad \phi = \mu$$

$$S_{YM} + S_{CS} = 2\mu^4\,\mathrm{tr}\left(\mathbb{1} - \frac{2\mu^2}{3}(X^2 + Y^2)\right) \sim -N^2 < 0$$