# *Insights from Higher Gauge Theory: the quest for Quantum Gravity with matter*

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# A sketch of the talk

- 3-group and 3-gauge theory
  - ↪ based on *R. Picken and J. Faria Martins*, arXiv:0907.2566.
- $3BF$ action
  - ↪ Models with relevant dynamics
    based on A. Miković and M. Vojinović, arXiv: 1110.4694, and *TR and M. Vojinović*, arXiv:1904.07566.
- Gauge symmetry of $3BF$ theory
  - ↪ $G$-,$H$-,$L$-,$M$-, and $N$-gauge transformations and diffeomorphisms, *TR and M. Vojinović*, arXiv: 2101.04049.
- Quantization of the topological $3BF$ theory
  - ↪ the state sum Z is an example of Porter's TQFT for $d = 4$ and $n = 3$ *T. Porter* (1998),
    based on *TR and M. Vojinović,* arXiv: 2201.02572.
  - ↪ The construction of the state sum $Z$ and a proof that the $3BF$ state sum is invariant under Pachner moves.
    *TR and M. Vojinović,* arXiv: 2201.02572.
  - ↪ This is a generalization of the state sum based on the classical $2BF$ action with the underlying 2-group structure
    *F. Girelli, H. Pfeiffer and E. M. Popescu*, arXiv:0708.3051.
- Conclusions

# INTRODUCTION

$\hookrightarrow$ The goal is to define the configurational integral $Z = \int \mathcal{D}\phi e^{iS[\phi]}$. On a discretized $D$-dimensional manifold, we define the **state sum**:

$$Z = \sum_{\{\phi\}} \prod_{v \in T} \mathcal{A}_v(\phi) \prod_{\epsilon \in T} \mathcal{A}_\epsilon(\phi) \cdots \prod_{\sigma \in T} \mathcal{A}_\sigma(\phi).$$

- The triangulation $\mathcal{T}(\mathcal{M}_D)$ of the manifold $\mathcal{M}_D$ contains vertices $v$, edges $\epsilon$, faces $\Delta$, tetrahedra $\tau$,...,$D$-simplices $\sigma$.
- Each of these simplices is colored with color $\phi$ - describing the fundamental variables of the model.
- To each simplex is assigned an amplitude $\mathcal{A}$ - describing the dynamics of variable $\phi$.

1. Write the action in the appropriate form:

$$S_{GR}[g] = S_{top}[g] + S_{constraints}[g].$$

2. Construct the topological state sum:

$$Z = \int \mathcal{D}g e^{iS_{top}}.$$

3. By modifying amplitudes in a certain way, we define the state sum corresponding to the complete theory:

$$Z = \int \mathcal{D}g e^{iS_{top} + iS_{constraints}}.$$

Theories of quantum gravity within the covariant approach are defined by quantizing the $BF$ theory with constraints for the Lie group $G$,

$$S_{BF} = \int_{\mathcal{M}^4} \langle B \wedge F \rangle_{\mathfrak{g}}, .$$

↪ *Ponzano-Regge model* of $3D$ gravity for the $SU(2)$ group. Ponzano and Regge, 1968.

↪ *Barrett-Crane model* of $4D$ gravity for $SO(3,1)$. Barrett and Crane, 1998.

↪ *EPRL/FK model* of $4D$ gravity, also known as the ***spinfoam model***.

J. Engle, R. Pereira, E. Livine, and K. Rovelli, and L. Freidel and K. Krasnov, 2008.

↪ All these models are focused on defining the *theory of pure gravity without matter*.

↪ Attempts to add matter fields into the theory have had limited success, mainly due to the fact that mass terms cannot be expressed within these theories (tetrad fields are not present in the topological sector of the $BF$ theory).

$\hookrightarrow$ In order to overcome the issue of matter coupling in BF models of quantum gravity, a new approach is developed within the framework of category theory, based on a categorical generalization of the BF action – the so-called $2BF$ action ($BFCG$ action).

$$S_{2BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge G \rangle_{\mathfrak{h}} \,.$$

$\hookrightarrow$ *Spin-cube model* of $4D$ gravity for the Poincaré 2-group. A. Miković and M. Vojinović, 2012.

$\hookrightarrow$ This result has opened the possibility of coupling matter with gravity in a linear fashion.

# Crossed Module

- In the framework of category theory, the group as an algebraic structure can be understood as a category with only one object and invertible morphisms
- The notion of a category can be generalized to the so-called higher categories, which have not only objects and morphisms, but also 2-morphisms (morphisms between morphisms), and so on.
- Similarly to the notion of a group, one can introduce a 2-group as a 2-category consisting of only one object, where all the morphisms and 2-morphisms are invertible.
- A 2-group is equivalent to a crossed module $(H \stackrel{\partial}{\to} G\,, \triangleright)$:
  - $G$ is a group with composition of morphisms as the group operation

$$\begin{tikzcd} \bullet & \bullet \arrow[l, bend right, "g"'] & \bullet \arrow[l, bend right, "g'"'] \end{tikzcd} = \begin{tikzcd} \bullet & \bullet \arrow[l, bend right, "gg'"'] \end{tikzcd} ,$$

  - $H$ is a group consisting of all 2-morphisms having the identity morphism as the source

$$\begin{tikzcd} \bullet & \bullet \arrow[l, bend right, "1_\bullet"', ""{name=A}] \arrow[l, bend left, "\partial h", ""'{name=B}] \arrow[from=A, to=B, Rightarrow, "h"] & \bullet \arrow[l, bend right, "1_\bullet"', ""{name=C}] \arrow[l, bend left, "\partial h'", ""'{name=D}] \arrow[from=C, to=D, Rightarrow, "h'"] \end{tikzcd} = \begin{tikzcd} \bullet & \bullet \arrow[l, bend right, "1_\bullet"', ""{name=A}] \arrow[l, bend left, "\partial(hh')", ""'{name=B}] \arrow[from=A, to=B, Rightarrow, "hh'"] \end{tikzcd} ,$$

  - Action of $G$ on $H$ given by the operation $\triangleright$: $G \to Aut(H)$

$$\begin{tikzcd} \bullet & \bullet \arrow[l, bend right, "g"', ""{name=A}] \arrow[l, bend left, "g", ""'{name=B}] \arrow[from=A, to=B, Rightarrow, "1_g"] & \bullet \arrow[l, bend right, "1_\bullet"', ""{name=C}] \arrow[l, bend left, "\partial h", ""'{name=D}] \arrow[from=C, to=D, Rightarrow, "h"] & \bullet \arrow[l, bend right, "g^{-1}"', ""{name=E}] \arrow[l, bend left, "g^{-1}", ""'{name=F}] \arrow[from=E, to=F, Rightarrow, "1_g^{-1}"] \end{tikzcd} = \begin{tikzcd} \bullet & \bullet \arrow[l, bend right, "1"', ""{name=A}] \arrow[l, bend left, "\partial(g\triangleright h)", ""'{name=B}] \arrow[from=A, to=B, Rightarrow, "g\triangleright h"] \end{tikzcd} ,$$

  - Group homomorphism $\partial : H \to G$

$$\begin{tikzcd} \bullet & \bullet \arrow[l, bend right, "1_\bullet"', ""{name=A}] \arrow[l, bend left, "\partial h", ""'{name=B}] \arrow[from=A, to=B, Rightarrow, "h"] \end{tikzcd} .$$

**Gravity**

A. Miković and M. Vojinović, arXiv: 1110.4694.

↪ Crossed module $(H \overset{\partial}{\to} G\,, \triangleright)$:

- $G = SO(3,1)\,, \qquad H = \mathbb{R}^4\,,$
- $M_{ab} \triangleright P_c = [M_{ab}, P_c]\,,$
- $\partial(\tau_\alpha) = 0\,.$

↪ 2-connection $(\alpha, \beta)$: $\boxed{\alpha = \omega^{ab} M_{ab}\,, \qquad \beta = \beta^a P_a\,.}$

↪ 2-curvature $(\mathcal{F}, \mathcal{G})$: $\boxed{\mathcal{F} = R^{ab} M_{ab}\,, \quad \mathcal{G} = \nabla \beta \, P_a\,.}$

↪ Topological action:

$$\boxed{S_{2BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla \beta^a\,.}$$

↪ Constrained action:

$$\boxed{S = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla \beta^a - \lambda_{ab} \wedge \left( B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \right).}$$

- Although the group structure is sufficient to describe gauge fields and the structure of 2-groups has been successfully applied to describe the gravitational field, *they are insufficient to describe other matter fields*, such as scalar and fermionic fields.
- To describe these fields, it is necessary to take another step in the **categorical ladder**, a categorical generalization of the algebraic structure of 2-groups to the structure of 3-groups.
- It turns out that the structure of 3-groups successfully describes all fields present in the Standard Model coupled to gravity.

| categorical structure | algebraic structure | linear structure | topological action | degrees of freedom |
|---|---|---|---|---|
| Lie group | Lie group | Lie algebra | $BF$ theory | gauge fields |
| Lie 2-group | Lie crossed module | differential Lie crossed module | $2BF$ theory | tetrad fields |
| Lie 3-group | Lie 2-crossed module | differential Lie 2-crossed module | $3BF$ theory | scalar and fermion fields |

**2-crossed module** $(L \overset{\delta}{\rightarrow} H \overset{\partial}{\rightarrow} G, \triangleright, \{\_, \_\}_{\mathrm{p}})$

↪ Groups $G$, $H$, and $L$;

↪ Mappings $\partial$ and $\delta$ ($\partial\delta = 1_G$);

↪ Action $\triangleright$ of group $G$ on all three groups;

↪ Mapping $\{\_, \_\}_{\mathrm{p}}$ – *Paiffer lifting*:

$$\{\_, \_\}_{\mathrm{p}} : H \times H \to L\,.$$

These groups and mappings must satisfy certain axioms in order to form a 2-crossed module:

1. $\delta(\{h_1, h_2\}_{\mathrm{p}}) = \langle h_1\,, h_2\rangle_{\mathrm{p}}, \qquad \forall h_1, h_2 \in H,$
2. $[l_1, l_2] = \{\delta(l_1)\,, \delta(l_2)\}_{\mathrm{p}}, \qquad \forall l_1\,, l_2 \in L$. Notation is $[l, k] = lkl^{-1}k^{-1}$;
3. $\{h_1h_2, h_3\}_{\mathrm{p}} = \{h_1, h_2h_3h_2^{-1}\}_{\mathrm{p}}\partial(h_1) \triangleright \{h_2, h_3\}_{\mathrm{p}}, \qquad \forall h_1, h_2, h_3 \in H;$
4. $\{h_1, h_2h_3\}_{\mathrm{p}} = \{h_1, h_2\}_{\mathrm{p}}\{h_1, h_3\}_{\mathrm{p}}\{\langle h_1, h_3\rangle_{\mathrm{p}}^{-1}, \partial(h_1) \triangleright h_2\}_{\mathrm{p}}, \qquad \forall h_1, h_2, h_3 \in H;$
5. $\{\delta(l), h\}_{\mathrm{p}}\{h, \delta(l)\}_{\mathrm{p}} = l(\partial(h) \triangleright l^{-1}), \qquad \forall h \in H\,, \quad \forall l \in L.$

3-group, i.e., 2-crossed module allows us to describe 3-gauge theory.

↪ The structure of 2-crossed module leads to ***3-connections***, ordered triples $(\alpha, \beta, \gamma)$, where $\alpha$, $\beta$, and $\gamma$ are differential form elements of algebras,

$$
\begin{aligned}
\alpha &= \alpha^{\alpha}{}_{\mu}\, \tau_{\alpha}\, \mathrm{d}x^{\mu}\,, & \alpha &\in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{g})\,,\\
\beta &= \beta^{a}{}_{\mu\nu}\, t_{a}\, \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu}\,, & \beta &\in \mathcal{A}^2(\mathcal{M}_4, \mathfrak{h})\,,\\
\gamma &= \gamma^{A}{}_{\mu\nu\rho}\, T_{A}\, \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu} \wedge \mathrm{d}x^{\rho}\,, & \gamma &\in \mathcal{A}^3(\mathcal{M}_4, \mathfrak{l})\,.
\end{aligned}
$$

↪ Then we define ***line, surface, and volume holonomies***,

$$
g = \exp \int_{\gamma} \alpha\,, \quad h = \exp \int_{S} \beta\,, \quad l = \exp \int_{V} \gamma\,.
$$

↪ The corresponding ***fake 3-curvature*** $(\mathcal{F}, \mathcal{G}, \mathcal{H})$ is defined as:

$$
\mathcal{F} = \mathrm{d}\alpha + \alpha \wedge \alpha - \partial\beta\,, \qquad \mathcal{G} = \mathrm{d}\beta + \alpha \wedge^{\triangleright} \beta - \delta\gamma\,,
$$
$$
\mathcal{H} = \mathrm{d}\gamma + \alpha \wedge^{\triangleright} \gamma + \{\beta \wedge \beta\}_{\mathrm{pf}}\,.
$$

↪ For a manifold $\mathcal{M}_4$ and a 2-crossed module $(L \xrightarrow{\delta} H \xrightarrow{\partial} G\,, \triangleright\,, \{\_\,,\_\}_{\mathrm{pf}})$, or 3-curvature $(\mathcal{F}, \mathcal{G}, \mathcal{H})$, the $3BF$ action is defined as:

$$S_{3BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G} \rangle_{\mathfrak{h}} + \langle D \wedge \mathcal{H} \rangle_{\mathfrak{l}}\,.$$

- $3BF$ theory is a topological theory,
- it relies on the structure of a 3-group,
- it's a generalization of the $BF$ topological theory based on the group structure $G$.

↪ Physical interpretation of Lagrange multipliers $C$ and $D$:

- 1-form $C$ with values in the algebra $\mathfrak{h}$ can be interpreted as the tetrad field if $H = \mathbb{R}^4$:

$$C \to e = e^a{}_\mu(x) t_a \mathrm{d}x^\mu\,,$$

A. Miković and M. Vojinović, arXiv: 1110.4694.

- function $D$ with values in the algebra $\mathfrak{l}$ can be interpreted as a set of real fields, for an appropriate choice of group $L$:

$$D \to \phi = \phi^A(x) T_A\,.$$

**2-crossed module for (trivial) Standard Model:**

- Groups

$$G = SO(3,1) \times SU(3) \times SU(2) \times U(1)\,, \quad H = \mathbb{R}^4\,, \quad L \; - \; \text{matter sector}\,;$$

- Mappings $\delta$ and $\partial$ are trivial — for all $l \in L$ and $\vec{v} \in H$, we define

$$\delta l = 1_H = 0\,, \quad \partial\vec{v} = 1_G\,;$$

- Paiffer lifting is trivial - for all $\vec{u}, \vec{v} \in H$, we define

$$\{\vec{u}, \vec{v}\}_{\mathrm{pf}} = 1_L\,;$$

- Action $\triangleright$ of group $G$ on itself is in the adjoint representation;
- Action $\triangleright$ of group $G$ on $H$ is in the vector representation for the $SO(3,1)$ sector and in the trivial representation for the $SU(3) \times SU(2) \times U(1)$ sector;
- Action of group $G$ on $L$ is non-trivial and depends on the choice of group $L$ - determines the transformation properties of fields.

**How do we choose the group $L$?**

↪ Since $\phi = \phi^A T_A$, we have one real field $\phi^A(x)$ for each generator of the group $L$.

↪ How many real fields are needed to describe the matter sector of the Standard Model?

| Lepton 1st generation | Red color<br>1st generation quarks | Green color<br>1st generation quarks | Blue color<br>1st generation quarks |
|---|---|---|---|
| $\begin{pmatrix}\nu_e \\ e^-\end{pmatrix}_L$ | $\begin{pmatrix}u_r \\ d_r\end{pmatrix}_L$ | $\begin{pmatrix}u_g \\ d_g\end{pmatrix}_L$ | $\begin{pmatrix}u_b \\ d_b\end{pmatrix}_L$ |
| $(\nu_e)_R$ | $(u_r)_R$ | $(u_g)_R$ | $(u_b)_R$ |
| $(e^-)_R$ | $(d_r)_R$ | $(d_g)_R$ | $(d_b)_R$ |

# Group $L$

$\hookrightarrow$ How many real components of fields do we have in the matter sector of the Standard Model?

- Fermion sector:

$$16\,\frac{\text{spinors}}{\text{family}} \times 3\,\text{families} \times 4\,\frac{\text{real fields}}{\text{spinor}} = 192\,\text{real fields}\,\phi^A.$$

- Higgs sector:

$$2\,\text{complex scalar fields} = 4\,\text{real fields}\,\phi^A.$$

- We obtain that the group structure $L$:

$$L = L_{fermion} \times L_{Higgs}\,, \quad \dim L_{fermion} = 192\,, \quad \dim L_{Higgs} = 4\,.$$

$\hookrightarrow$ The action $G \triangleright L \to L$ determines the <u>transformation properties</u> of the real fields $\phi^A$ under Lorentz and internal transformations.

$\hookrightarrow$ $G$ acts in the same way in each family, so the group $L$ has the structure:

$$L_{fermion} = L_{1st\,family} \times L_{2nd\,family} \times L_{3rd\,family}\,, \quad \dim L_{k-th\,family} = 64\,.$$

- **The action $G \triangleright L \to L$ determines the transformation properties of the real fields $\phi^A$ under Lorentz and internal transformations.**

  For example, consider a doublet $\begin{pmatrix} u_b \\ d_b \end{pmatrix}_L$. The action $g \triangleright u_b$ encodes that $u_b$ consists of 4 real-valued fields which transform as:

  - a left-handed spinor wrt. $SO(3,1)$,
  - as a “blue” component of the fundamental representation of $SU(3)$,
  - and as ”isospin $+\frac{1}{2}$” of the left doublet wrt. $SU(2) \times U(1)$.

- The structure of 3-groups successfully provides a description of all fields present in the Standard Model, interacting with gravity.
- Additionally, this structure naturally associates a *new gauge group to the* ***scalar*** *and* ***fermionic*** *fields present in the theory*, thus generalizing the concept of gauge groups in Yang-Mills theory.
- After determining the appropriate 3-groups and constructing the corresponding $3BF$ actions, it is necessary to impose appropriate **constraints** on the degrees of freedom present in the topological sector of the $3BF$ action, in order to obtain the desired classical dynamics of matter and gravity fields.

↪ For the manifold $\mathcal{M}_4$ and the 2-crossed module $(L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G\,, \triangleright\,, \{\_\,,\_\}_{\mathrm{pf}})$, or equivalently for the 3-curvature $(\mathcal{F}, \mathcal{G}, \mathcal{H})$, the $3BF$ action is defined as:

$$S_{3BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G} \rangle_{\mathfrak{h}} + \langle D \wedge \mathcal{H} \rangle_{\mathfrak{l}}\,.$$

↪ By adding constraints to the topological action, physically relevant models are defined:

$2BF$ **actions with constraints** for:

- *Yang-Mills field,*
- and *Einstein-Cartan gravity,*

and $3BF$ **actions with constraints** describing

- *Klein-Gordon field,*
- *Dirac field,*
- *Weyl field,*
- and *Majorana field.*

coupled to gravity in the standard manner.

**Gravity and $SU(N)$ Yang-Mills Field**

TR and M. Vojinović, arXiv: 1904.07566.

↪ Crossed module $(H \overset{\partial}{\to} G\,,\triangleright)$:

- $G = SO(3,1)\times SU(N)\,, \qquad H = \mathbb{R}^4\,,$
- $M_{ab} \triangleright P_c = [M_{ab}, P_c]\,, \quad \tau_I \triangleright P_a = 0\,,$
- $\partial(\tau_I) = 0\,.$

↪ 2-connection $(\alpha,\beta)$: $\boxed{\alpha = \omega^{ab}M_{ab} + A^I\tau_I\,, \qquad \beta = \beta^a P_a\,.}$

↪ 2-curvature $(\mathcal{F},\mathcal{G})$: $\boxed{\mathcal{F} = R^{ab}M_{ab} + F^I\tau_I\,, \quad \mathcal{G} = \nabla\beta P_a\,.}$

↪ Topological action: $\boxed{S_{2BF} = \int_{\mathcal{M}_4} B^{ab}\wedge R_{ab} + B^I\wedge F_I + e_a\wedge\nabla\beta^a\,.}$

↪ Constrained action:

$$\boxed{\begin{aligned} S = \int_{\mathcal{M}_4} & B^{ab}\wedge R_{ab} + B^I\wedge F_I + e_a\wedge\nabla\beta^a - \lambda_{ab}\wedge\left(B^{ab} - \frac{1}{16\pi l_p^2}\varepsilon^{abcd}e_c\wedge e_d\right) \\ & + \lambda^I\wedge\left(B_I - \frac{12}{g}M_{ab\,I}e^a\wedge e^b\right) + \zeta^{ab\,I}\left(M_{ab\,I}\varepsilon_{cdef}e^c\wedge e^d\wedge e^e\wedge e^f - g_{IJ}F^J\wedge e_a\wedge e_b\right). \end{aligned}}$$

**Klein-Gordon field** $\boxed{D = \phi \mathbb{I}}$

TR and M. Vojinović, arXiv: 1904.07566.

↪ 2-crossed module $(L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G \,, \triangleright \,, \{\_\,,\_\})$ :

- $G = SO(3,1)\,, \qquad H = \mathbb{R}^4\,, \qquad L = \mathbb{R}\,,$
- $M_{ab} \triangleright P_c = [M_{ab}, P_c]\,, \qquad M_{ab} \triangleright T_A = 0\,,$
- $\partial(P_a) = 0\,, \qquad \delta(T_A) = 0\,, \qquad \{P_a\,, P_b\} = 0\,.$

↪ 3-connection $(\alpha\,,\beta\,,\gamma)$: $\boxed{\alpha = \omega^{ab} M_{ab}\,, \qquad \beta = \beta^a P_a\,, \qquad \gamma = \gamma \mathbb{I}\,.}$

↪ 3-curvature $(\mathcal{F}\,,\mathcal{G}\,,\mathcal{H})$: $\boxed{\mathcal{F} = R^{ab} M_{ab}\,, \qquad \mathcal{G} = \nabla \beta^a P_a\,, \qquad \mathcal{H} = \mathrm{d}\gamma\,.}$

↪ Topological action: $\boxed{S_{3BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla \beta^a + \phi \, \mathrm{d}\gamma\,.}$

$$
\boxed{\begin{aligned}
S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + e_a \wedge \nabla \beta^a + \phi \, \mathrm{d}\gamma - \lambda_{ab} \wedge \left( B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \right) \\
& + \lambda \wedge \left( \gamma - \frac{1}{2} H_{abc} e^a \wedge e^b \wedge e^c \right) + \Lambda^{ab} \wedge \left( H_{abc} \varepsilon^{cdef} e_d \wedge e_e \wedge e_f - \mathrm{d}\phi \wedge e_a \wedge e_b \right) \\
& - \frac{1}{2 \cdot 4!} m^2 \phi^2 \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d\,.
\end{aligned}}
$$

- General relativity can be formulated as a $2BF$ theory with constraints for a specific choice of a symmetry 2-group. A. Miković and M. Vojinović, arXiv: 1110.4694.
  - The advantage of this formulation of General Relativity over the formulation via $BF$ theory lies in the fact that the structure of the 2-group introduces tetrad fields into the topological action, allowing for matter coupling with gravity in a straightforward manner.
  - However, matter fields cannot be naturally expressed within the algebraic structure of the 2-group, i.e., the matter sector in the action cannot be written as a sum of a topological term and constraint term.
  - Another step of higher categorical generalization of the $BF$ theory is necessary — the so-called $3BF$ theory.
- The *Einstein-Yang-Mills theories* have been formulated, i.e., the theory of gravity and gauge fields as $2BF$ theory with constraints.
- Theories describing the *Klein-Gordon* and *Dirac fields* in curved space have been formulated as $3BF$ action with constraints, as well as the *Weyl* and *Majorana fields* interacting with Einstein-Cartan gravity.
- These results are then applied to construct $3BF$ actions with constraints describing all matter present in the Standard Model coupled to the gravitational field.
  - The advantage of this formulation lies in having the classical action of the complete theory written in a form prepared for the spinfoam quantization procedure.

TR and M. Vojinović, arXiv: 1904.07566.

# Gauge symmetry in $3BF$ theory

## $G$-gauge transformations

J. F. Martins and R. Picken, 2011, W. Wang, 2014.

In the $3BF$ theory for a 2-crossed module $(L \xrightarrow{\delta} H \xrightarrow{\partial} G, \rhd, \{\_\,,\_\}_{\mathrm{pf}})$, the following transformation is a gauge symmetry:

$$
\begin{array}{cclccl}
\alpha & \to & \alpha' = \mathrm{Ad}_g \alpha + g dg^{-1} \,, & B & \to & B' = gBg^{-1} \,, \\
\beta & \to & \beta' = g \rhd \beta \,, & C & \to & C' = g \rhd C \,, \\
\gamma & \to & \gamma' = g \rhd \gamma \,, & D & \to & D' = g \rhd D \,,
\end{array}
$$

where $g = \exp(\epsilon_{\mathfrak{g}} \cdot \hat{G}) = \exp(\epsilon_{\mathfrak{g}\,\alpha} \hat{G}^{\alpha}) \in G$ and $\epsilon_{\mathfrak{g}} : \mathcal{M}_4 \to \mathfrak{g}$ is the transformation parameter.

## $H$-gauge transformation

J. F. Martins and R. Picken, 2011, W. Wang, 2014.

↪ In the $3BF$ theory for a 2-crossed module $(L \xrightarrow{\delta} H \xrightarrow{\partial} G, \rhd, \{\_\,,\_\}_{\mathrm{pf}})$, the following transformation is a symmetry:

$$
\begin{array}{cclccl}
\alpha & \to & \alpha' = \alpha - \partial \epsilon_{\mathfrak{h}} \,, & B & \to & B' = B - C' \wedge^{\mathcal{T}} \epsilon_{\mathfrak{h}} - \epsilon_{\mathfrak{h}} \wedge^{\mathcal{D}} \epsilon_{\mathfrak{h}} \wedge^{\mathcal{D}} D \,, \\
\beta & \to & \beta' = \beta - \nabla' \epsilon_{\mathfrak{h}} - \epsilon_{\mathfrak{h}} \wedge \epsilon_{\mathfrak{h}} \,, & C & \to & C' = C - D \wedge^{\mathcal{X}_1} \epsilon_{\mathfrak{h}} - D \wedge^{\mathcal{X}_2} \epsilon_{\mathfrak{h}} \,, \\
\gamma & \to & \gamma' = \gamma + \{\beta', \epsilon_{\mathfrak{h}}\}_{\mathrm{pf}} + \{\epsilon_{\mathfrak{h}}, \beta\}_{\mathrm{pf}} \,, & D & \to & D' = D \,,
\end{array}
$$

where $\epsilon_{\mathfrak{h}} \in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{h})$ is an arbitrary 1-form element of the algebra $\mathfrak{h}$.

## $L$-gauge transformations

J. F. Martins and R. Picken, 2011., W. Wang, 2014.

↪ In the $3BF$ theory for a 2-crossed module $(L \xrightarrow{\delta} H \xrightarrow{\partial} G, \rhd, \{\_\,,\_\}_{\mathrm{pf}})$, the following transformation is a symmetry:

$$
\begin{array}{cclccl}
\alpha & \to & \alpha' = \alpha \,, & B & \to & B' = B + D \wedge^{\mathcal{S}} \epsilon_{\mathfrak{l}} \,, \\
\beta & \to & \beta' = \beta + \delta \epsilon_{\mathfrak{l}} \,, & C & \to & C' = C \,, \\
\gamma & \to & \gamma' = \gamma + \nabla \epsilon_{\mathfrak{l}} \,, & D & \to & D' = D \,,
\end{array}
$$

where $\epsilon_{\mathfrak{l}} \in \mathcal{A}^2(\mathcal{M}_4, \mathfrak{l})$ is an arbitrary 2-form element of the algebra $\mathfrak{l}$.

TR and M. Vojinović, arXiv: 2101.04049.

### $M$-gauge transformations

↪ In the 3$BF$ theory for a 2-crossed module $(L \overset{\delta}{\to} H \overset{\partial}{\to} G, \triangleright, \{\_\,,\_\}_{\mathrm{pf}})$, the following transformation is a symmetry

$$
\begin{aligned}
\alpha &\to \alpha' = \alpha\,, & B &\to B' = B - \nabla\epsilon_{\mathfrak{m}}\,,\\
\beta &\to \beta' = \beta\,, & C^a &\to C'^a = C^a - \partial^a{}_\alpha \epsilon_{\mathfrak{m}}{}^\alpha\,,\\
\gamma &\to \gamma' = \gamma\,, & D &\to D' = D\,,
\end{aligned}
$$

where $\epsilon_{\mathfrak{m}} \in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{g})$ is an arbitrary 1-form element of the algebra $\mathfrak{g}$.

### $N$-gauge transformations

↪ In the 3$BF$ theory for a 2-crossed module $(L \overset{\delta}{\to} H \overset{\partial}{\to} G, \triangleright, \{\_\,,\_\}_{\mathrm{pf}})$, the following transformation is a symmetry

$$
\begin{aligned}
\alpha &\to \alpha' = \alpha\,, & B &\to B' = B - \beta \wedge^{\mathcal{T}} \epsilon_{\mathfrak{n}}\,,\\
\beta &\to \beta' = \beta\,, & C &\to C' = C - \nabla\epsilon_{\mathfrak{n}}\,,\\
\gamma &\to \gamma' = \gamma\,, & D^A &\to D'^A = D^A + \delta^A{}_a \epsilon_{\mathfrak{n}}{}^a\,,
\end{aligned}
$$

where $\epsilon_{\mathfrak{n}} : \mathcal{M}_4 \to \mathfrak{h}$ is an arbitrary function element of the algebra $\mathfrak{h}$.

↪ We obtain that the Lie algebra $\mathfrak{g}$ of the group $G$ of a 2-crossed module $(L \overset{\delta}{\to} H \overset{\partial}{\to} G\,, \triangleright\,, \{\_\,,\_\}_{\mathrm{pf}})$ is:

$$[\hat{G}_\alpha, \hat{G}_\beta] = f_{\alpha\beta}{}^{\gamma}\hat{G}_\gamma\,.$$

↪ Algebra of the group $\tilde{H}_L$ (generators of $H$- and $L$-gauge transformations):

$$[\hat{H}_a{}^{\mu}, \hat{H}_b{}^{\nu}] = 2X_{(ab)}{}^{A}\hat{L}_A{}^{\mu\nu}\,, \qquad [\hat{L}_A{}^{\mu\nu}, \hat{L}_B{}^{\rho\sigma}] = 0\,, \qquad [\hat{H}_a{}^{\mu}, \hat{L}_A{}^{\nu\rho}] = 0\,.$$

↪ Groups $\tilde{M}$ and $\tilde{N}$ (generators of $M$-gauge transformations and $N$-gauge transformations)

$$[\hat{M}_\alpha{}^{\mu}, \hat{M}_\beta{}^{\nu}] = 0\,, \quad [\hat{N}_a, \hat{N}_b] = 0\,, \quad [\hat{M}_\alpha{}^{\mu}, \hat{N}_a] = 0\,.$$

↪ Action of generators of the group $\tilde{H}_L$ on generators of $M$- and $N$-gauge transformations:

$$\begin{aligned} [\hat{H}_a{}^{\mu}, \hat{N}^b] &= \triangleright_{\alpha a}{}^{b}\hat{M}^{\alpha\mu}\,, & [\hat{H}_a{}^{\mu}, \hat{M}_\alpha{}^{\nu}] &= 0\,, \\ [\hat{L}_A{}^{\nu\rho}, \hat{M}_\alpha{}^{\mu}] &= 0\,, & [\hat{L}_A{}^{\mu\nu}, \hat{N}_a] &= 0\,. \end{aligned}$$

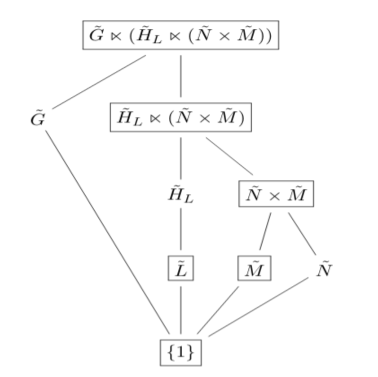
↪ Action of generators of group $G$ on generators of $H$-, $L$-, $M$- and $N$-gauge transformations:

$$\begin{aligned} [\hat{G}_\alpha, \hat{H}_a{}^{\mu}] &= \triangleright_{\alpha a}{}^{b}\hat{H}_b{}^{\mu}\,, & [\hat{G}_\alpha, \hat{L}_A{}^{\mu\nu}] &= \triangleright_{\alpha A}{}^{B}\hat{L}_B{}^{\mu\nu}\,, \\ [\hat{G}_\alpha, \hat{M}_\beta{}^{\mu}] &= f_{\alpha\beta}{}^{\gamma}\hat{M}_\gamma{}^{\mu}\,, & [\hat{G}_\alpha, \hat{N}_a] &= \triangleright_{\alpha a}{}^{b}\hat{N}_b\,. \end{aligned}$$

$\hookrightarrow$ Summarizing the previous results, we find that the gauge symmetry group $\mathcal{G}_{3BF}$ has the structure:

$$\mathcal{G}_{3BF} = \tilde{G} \ltimes (\tilde{H}_L \ltimes (\tilde{N} \times \tilde{M}))\,.$$

TR and M. Vojinović, arXiv: 2101.04049.



$\hookrightarrow$ Any action depending on at least two fields $\phi_1(x)$ and $\phi_2(x)$ is invariant under the following transformation, determined by the HT parameter $\epsilon^{\mathrm{HT}}$:

$$\delta_0{}^{\mathrm{HT}}\phi_1 = \epsilon^{\mathrm{HT}}(x)\frac{\delta S}{\delta\phi_2}\,, \qquad \delta_0{}^{\mathrm{HT}}\phi_2 = -\epsilon^{\mathrm{HT}}(x)\frac{\delta S}{\delta\phi_1}\,,$$

which can be easily verified by calculating the variation of the action:

$$\delta^{\mathrm{HT}}S[\phi_1,\phi_2] = \frac{\delta S}{\delta\phi_1}\delta_0{}^{\mathrm{HT}}\phi_1 + \frac{\delta S}{\delta\phi_2}\delta_0{}^{\mathrm{HT}}\phi_2 = 0\,.$$

$\hookrightarrow$ **If diffeomorphisms are symmetries of the action**, then for every field $\phi(x)$ in the theory, and every parameter of diffeomorphisms $\xi^\mu(x)$, **there exists a choice of parameters** $\epsilon_i(x)$ and $\epsilon^{\mathrm{HT}}(x)$, such that:

$$(\delta_0{}^{\mathrm{gauge}} + \delta_0{}^{\mathrm{HT}} + \delta_0{}^{\mathrm{diff}})\,\phi = 0\,.$$

If diffeomorphisms are symmetries of the theory, their variation of the form can be expressed in terms of variations of the form corresponding to gauge and HT transformations:

$$\delta_0{}^{\mathrm{diff}}\,\phi = -\delta_0{}^{\mathrm{gauge}}\phi - \delta_0{}^{\mathrm{HT}}\phi\,.$$

*"Quantization of gauge systems"*, Eno and Teitelboim, 1994.

↪ HT variations of forms are defined as:

$$
\begin{aligned}
\delta_0{}^{\mathrm{HT}}\alpha^\alpha{}_\mu &= \frac{1}{2}\epsilon^{\mathrm{HT}\,\alpha\beta}{}_{\mu\nu\rho}\frac{\delta S}{\delta B^\beta{}_{\nu\rho}}\,, & \delta_0{}^{\mathrm{HT}}B^\alpha{}_{\mu\nu} &= -\epsilon^{\mathrm{HT}\,\alpha\beta}{}_{\rho\mu\nu}\frac{\delta S}{\delta\alpha^\beta{}_\rho}\,,\\
\delta_0{}^{\mathrm{HT}}\beta^a{}_{\mu\nu} &= \epsilon^{\mathrm{HT}\,ab}{}_{\mu\nu\rho}\frac{\delta S}{\delta C^b{}_\rho}\,, & \delta_0{}^{\mathrm{HT}}C^a{}_\mu &= -\frac{1}{2}\epsilon^{\mathrm{HT}\,ab}{}_{\nu\rho\mu}\frac{\delta S}{\delta\beta^b{}_{\nu\rho}}\,,\\
\delta_0{}^{\mathrm{HT}}\gamma^A{}_{\mu\nu\rho} &= \epsilon^{\mathrm{HT}\,AB}{}_{\mu\nu\rho}\frac{\delta S}{\delta D^B}\,, & \delta_0{}^{\mathrm{HT}}D^A &= -\frac{1}{3!}\epsilon^{\mathrm{HT}\,AB}{}_{\mu\nu\rho}\frac{\delta S}{\delta\gamma^B{}_{\mu\nu\rho}}\,,
\end{aligned}
$$

↪ The parameters of HT transformations are $\epsilon^{\mathrm{HT}\,\alpha\beta}{}_{\mu\nu\rho}\,, \epsilon^{\mathrm{HT}\,ab}{}_{\mu\nu\rho}$, and $\epsilon^{\mathrm{HT}\,AB}{}_{\mu\nu\rho}$.
↪ The parameters of gauge transformations are $\epsilon_{\mathfrak{g}}{}^\alpha$, $\epsilon_{\mathfrak{h}}{}^a{}_\mu$, $\epsilon_{\mathfrak{l}}{}^A{}_{\mu\nu}$, $\epsilon_{\mathfrak{m}}{}^\alpha{}_\mu$, and $\epsilon_{\mathfrak{n}}{}^a$.

There is a choice that gives diffeomorphisms!

$$
\boxed{\epsilon_{\mathfrak{g}}{}^\alpha = -\xi^\lambda\alpha^\alpha{}_\lambda\,, \quad \epsilon_{\mathfrak{h}}{}^a{}_\mu = \xi^\lambda\beta^a{}_{\mu\lambda}\,, \quad \epsilon_{\mathfrak{l}}{}^A{}_{\mu\nu} = \xi^\lambda\gamma^A{}_{\mu\nu\lambda}\,, \quad \epsilon_{\mathfrak{m}}{}^\alpha{}_\mu = \xi^\lambda B^\alpha{}_{\mu\lambda}\,, \quad \epsilon_{\mathfrak{n}}{}^a = -\xi^\lambda C^a{}_\lambda\,,}
$$

$$
\boxed{\epsilon^{\mathrm{HT}\,\alpha\beta}{}_{\mu\nu\rho} = \xi^\lambda g^{\alpha\beta}\epsilon_{\mu\nu\rho\lambda}\,, \qquad \epsilon^{\mathrm{HT}\,ab}{}_{\mu\nu\rho} = \xi^\lambda g^{ab}\epsilon_{\lambda\mu\nu\rho}\,, \qquad \epsilon^{\mathrm{HT}\,AB}{}_{\mu\nu\rho} = \xi^\lambda g^{AB}\epsilon_{\mu\nu\rho\lambda}\,,}
$$

↪ Hence, $3BF$ theory is invariant under diffeomorphism transformations.

↪ Diffeomorphisms are a subgroup of the semidirect product of the total gauge symmetry group $\mathcal{G}_{3BF}$ and the HT transformation group $\mathcal{G}_{HT}$.

$$
Diff(\mathcal{M}_4) \not\subset \mathcal{G}_{3BF}\,, \quad \text{but} \quad Diff(\mathcal{M}_4) \subset \mathcal{G}_{total} = \mathcal{G}_{3BF} \ltimes \mathcal{G}_{HT}\,.
$$



TR and M. Vojinović, arXiv: 2101.04049.

- After the Hamiltonian analysis of the theory, computing the generators using the Castellani procedure, and calculating their commutators, it was found that the $3BF$ theory is invariant under **five types of gauge transformations** — **$G$-gauge, $H$-gauge, $L$-gauge, $M$-gauge, and $N$-gauge transformations**.
- We analyzed the structure of the complete gauge symmetry group $\mathcal{G}_{3BF}$ – a **connection between the gauge symmetry group of the $3BF$ action and the structure of the 3-group** on which the $3BF$ action is based was obtained.
- As expected, it is established that the $3BF$ theory has **diffeomorphism symmetry**.

TR and M. Vojinović, arXiv: 2101.04049.

- The explicit symmetry breaking of the gauge group of the topological $3BF$ sector, due to the presence of the constraints, has been studied. Each constraint was studied separately, and it is analyzed which gauge sector is being broken by which constraint. P. Stipsić and M. Vojinović, arXiv: 2402.17675.
- In addition, the spontaneous symmetry breaking and the Higgs mechanism for the $3BF$ formulation of the electroweak model has been studied. While the Higgs mechanism is conceptually the same as in the ordinary electroweak theory, the structure and details of the $3BF$ formulation are very different from the standard textbook approach, so much that the complete procedure of spontaneous symmetry breaking had to be done anew. P. Stipsić and M. Vojinović, arXiv: 2402.17675.

# CONSTRUCTION OF $3BF$ STATE SUM

Construction of the topological $3BF$ *state sum* based on the $S_{3BF}$ action using the standard *spinfoam quantization procedure*.

$$Z = \int \mathcal{D}\alpha\,\mathcal{D}\beta\,\mathcal{D}\gamma\,\mathcal{D}B\,\mathcal{D}C\,\mathcal{D}D \exp\left( i \int_{M_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G} \rangle_{\mathfrak{h}} + \langle D \wedge \mathcal{H} \rangle_{\mathfrak{l}} \right).$$

↪ By formally integrating over the Lagrange multipliers $B$, $C$, and $D$, we obtain:

$$Z = \mathcal{N} \int \mathcal{D}\alpha\,\mathcal{D}\beta\,\mathcal{D}\gamma\ \delta(\mathcal{F})\delta(\mathcal{G})\delta(\mathcal{H}).$$

↪ Discretization of the 3-connection:

- $\alpha \in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{g}) \mapsto g_\epsilon \in G$ colors the edges $\epsilon = (jk) \in \Lambda_1$,
- $\beta \in \mathcal{A}^2(\mathcal{M}_4, \mathfrak{h}) \mapsto h_\Delta \in H$ colors the triangles $\Delta = (jk\ell) \in \Lambda_2$,
- $\gamma \in \mathcal{A}^3(\mathcal{M}_4, \mathfrak{l}) \mapsto l_\tau \in L$ colors the tetrahedra $\tau = (jk\ell m) \in \Lambda_3$.

$$\left.\begin{array}{lcl} \int \mathcal{D}\alpha & \mapsto & \prod_{(jk)\in\Lambda_1} \int_G dg_{jk} \\ \int \mathcal{D}\beta & \mapsto & \prod_{(jk\ell)\in\Lambda_2} \int_H dh_{jk\ell} \\ \int \mathcal{D}\gamma & \mapsto & \prod_{(jk\ell m)\in\Lambda_3} \int_L dl_{jk\ell m} \end{array}\right\} \longrightarrow \text{Discretization of the integral measure.}$$

↪ The condition $\delta(\mathcal{F})$ is discretized as

$$\delta(\mathcal{F}) = \prod_{(jk\ell)\in\Lambda_2} \delta_G(g_{jk\ell})\,, \qquad \boxed{\delta_G(g_{jk\ell}) = \delta_G\big(\partial(h_{jk\ell})\, g_{k\ell}\, g_{jk}\, g_{j\ell}^{-1}\big)\,.}$$

↪ The condition $\delta(\mathcal{G})$ is discretized as

$$\delta(\mathcal{G}) = \prod_{(jk\ell m)\in\Lambda_3} \delta_H(h_{jk\ell m})\,,$$

$$\boxed{\delta_H(h_{jk\ell m}) = \delta_H\big(\delta(l_{jk\ell m}) h_{j\ell m}\, (g_{\ell m} \triangleright h_{jk\ell})\, h_{k\ell m}^{-1}\, h_{jkm}^{-1}\big)\,.}$$

↪ The condition $\delta(\mathcal{H})$ is discretized as

$$\delta(\mathcal{H}) = \prod_{(jk\ell mn)\in\Lambda_4} \delta_L(l_{jk\ell mn})\,,$$

$$\boxed{\delta_L(l_{jk\ell mn}) = \delta_L\big(l_{j\ell mn}^{-1}\, h_{j\ell n} \triangleright' \{h_{\ell mn}, (g_{mn} g_{\ell m}) \triangleright h_{jk\ell}\}_{\mathrm{p}}\, l_{jk\ell n}^{-1} (h_{jkn} \triangleright' l_{k\ell mn}) l_{jkmn} h_{jmn} \triangleright' (g_{mn} \triangleright l_{jk\ell m})\big)\,.}$$

...we obtain $\Longrightarrow$

$$\boxed{Z = \mathcal{N} \prod_{(jk)\in\Lambda_1} \int_G dg_{jk} \prod_{(jk\ell)\in\Lambda_2} \int_H dh_{jk\ell} \prod_{(jk\ell m)\in\Lambda_3} \int_L dl_{jk\ell m} \Big(\prod_{(jk\ell)\in\Lambda_2} \delta_G(g_{jk\ell})\Big)\Big(\prod_{(jk\ell m)\in\Lambda_3} \delta_H(h_{jk\ell m})\Big)\Big(\prod_{(jk\ell mn)\in\Lambda_4} \delta_L(l_{jk\ell mn})\Big).}$$

This expression becomes independent of the manifold triangulation by appropriate choice of the factor $\mathcal{N}$.

Let $\mathcal{M}_d$ be a compact oriented combinatorial $d$-manifold, $d = 4$, and let $(L \overset{\delta}{\to} H \overset{\partial}{\to} G\,, \triangleright\,, \{\_\,,\_\}_{\mathrm{pf}})$ be a 2-crossed module. The state sum of the topological 3-gauge theory is defined by the following expression:

$$
\begin{aligned}
Z \quad = \quad & |G|^{-|\Lambda_0|+|\Lambda_1|-|\Lambda_2|}|H|^{|\Lambda_0|-|\Lambda_1|+|\Lambda_2|-|\Lambda_3|}\,|L|^{-|\Lambda_0|+|\Lambda_1|-|\Lambda_2|+|\Lambda_3|-|\Lambda_4|} \\
& \times\Bigg(\prod_{(jk)\in\Lambda_1}\int_G dg_{jk}\Bigg)\Bigg(\prod_{(jk\ell)\in\Lambda_2}\int_H dh_{jk\ell}\Bigg)\Bigg(\prod_{(jk\ell m)\in\Lambda_3}\int_L dl_{jk\ell m}\Bigg) \\
& \times\Bigg(\prod_{(jk\ell)\in\Lambda_2}\delta_G\big(\partial(h_{jk\ell})\,g_{k\ell}\,g_{jk}\,g_{j\ell}^{-1}\big)\Bigg)\Bigg(\prod_{(jk\ell m)\in\Lambda_3}\delta_H\big(\delta(l_{jk\ell m})h_{j\ell m}\,(g_{\ell m}\triangleright h_{jk\ell})\,h_{k\ell m}^{-1}\,h_{jkm}^{-1}\big)\Bigg) \\
& \times\Bigg(\prod_{(jk\ell mn)\in\Lambda_4}\delta_L\big(l_{j\ell mn}^{-1}\,h_{j\ell n}\triangleright'\{h_{\ell mn},(g_{mn}g_{\ell m})\triangleright h_{jk\ell}\}_{\mathrm{p}}\,l_{jk\ell n}^{-1}(h_{jkn}\triangleright' l_{k\ell mn})l_{jkmn}h_{jmn}\triangleright'(g_{mn}\triangleright l_{jk\ell m})\big)\Bigg).
\end{aligned}
$$

Where $|\Lambda_0|$ denotes the number of vertices, $|\Lambda_1|$ the number of edges, $|\Lambda_2|$ triangles, $|\Lambda_3|$ tetrahedra, and $|\Lambda_4|$ the number of 4-simplices in the triangulation.

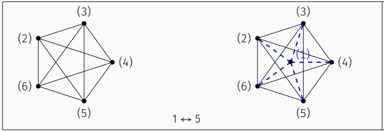
↪ *TR and M. Vojinović,* arXiv: 2201.02572.

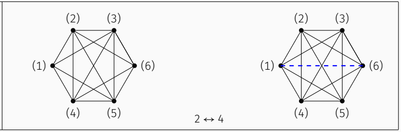
↪ Behavior under Pachner moves has been analyzed.

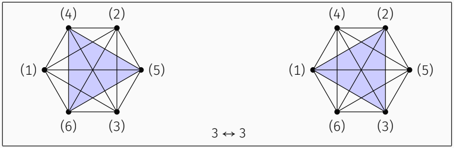
# Pachner moves

- We analyzed the behavior of the constructed state sum under Pachner moves.
  Pachner moves are local changes to triangulations that preserve topology, so any two triangulations of the same manifold are connected by a finite number of Pachner moves.
- **In the $3D$ case, there are four Pachner moves** — moves $1 \leftrightarrow 4$ and $2 \leftrightarrow 3$ and their inverses, while **in $4D$ there are five distinct Pachner moves** — moves $3 \leftrightarrow 3$, $4 \leftrightarrow 2$, and $5 \leftrightarrow 1$ and their inverses.

Pachner moves in $4D$



$2BF$ topological state sum

- We construct the $2BF$ action for a general strict 2-group and any triangulation of any smooth $d$-dimensional spacetime manifold, $d \in \{3, 4\}$.
  - For $d = 3$, the constructed state sum is precisely the Jetter's model.
  - For $d = 4$, it coincides with Porter's TQFT for $d = 4$ and $n = 2$.
- $2BF$ state sum is a *topological invariant* of the manifold.
  - *Girelli, Pfeiffer, Popescu,* arXiv: 0708.3051. *Miković, Martins,* arXiv: 1006.0903.
- Representation theory for 2-groups (including the Poincare 2-group), has been developed in great detail. *Baez, Baratin, Freidel, Wise* arXiv: 0812.4969.
- The topological invariant and TQFT for the Euclidean 2-group $(G = SO(4), H = \mathbb{R}^4)$ has also been studied in detail. *Asante, Dittrich, Girelli, Riello, Tsimiklis* arXiv: 1908.05970.

$2BF$ state sum

- For Poincare 2-group and $2BF$ action for GR, one possible quantization prescription leads to the *spincube model*. *A. Miković and M. Vojinović,* arXiv: 1110.4694.

$3BF$ topological state sum

- We formulate the $3BF$ state sum for the classical $3BF$ action in the case of a general semi-strict 3-group and 4-dimensional spacetime manifold.
  - It matches Porter's abstract definition of TQFT for $d = 4$ and $n = 3$.
- We find that it is a *topological invariant* of the manifold. *TR and M. Vojinović,* arXiv: 2201.02572.

- The state sum for the $3BF$ topological theory is obtained.
- However, to complete the **second step of the covariant spinfoam quantization procedure**, it is necessary to have generalizations of the Peter-Weyl and Plancherel theorems for the cases of 2-groups and 3-groups, mathematical results that are currently open problems.
- These theorems should **provide a decomposition of functions on a 3-group** into a sum over the corresponding irreducible representations of the 3-group.
- This determines the **spectrum of labels of the simplices of the triangulation**, i.e., the range of values of fields living on the simplices of the triangulation, as was done in the case of the $BF$ state sum.
- Current attempts of the second step of quantization of generalized $BF$ theories in the framework of higher gauge theories boil down to guessing irreducible representations of 2-groups.
- This result opens a way to **the third and final step of the covariant quantization procedure and the formulation of the quantum theory of gravity and matter of the Standard Model** by imposing appropriate constraints on the variables through modification of the amplitudes of the state sum.

- **First step of the covariant spinfoam quantization procedure. Classical theory.** — Successfully formulated constrained $3BF$ actions describing gravitational and Yang-Mills, scalar, and Dirac fields.
- **Gauge group of symmetries of the topological $3BF$ action.** — Complete Hamiltonian analysis of the $3BF$ action was performed, and the generator of gauge transformations was found. It was obtained that the $3BF$ theory is invariant under *five types of gauge transformations*: *$G$-gauge, $H$-gauge, $L$-gauge, $M$-gauge, and $N$-gauge transformations*.
- **Second step of the covariant spinfoam quantization procedure.** — Constructed the $3BF$ **state sum** and proved its invariance under Pachner moves, i.e., that it is a topological invariant of the manifold.
- **Third step of the covariant spinfoam quantization procedure.** — Work in progress!
- Nontrivial choices of the 3-group structure may provide new avenues for research on unification of all fields.

*Thank you for your attention!*

**Dirac Field** $\boxed{D = \psi^\alpha P_\alpha + \bar{\psi}_\alpha P^\alpha}$

↪ 2-crossed module $(L \xrightarrow{\delta} H \xrightarrow{\partial} G\,, \triangleright\,, \{\_\,,\_\})$ :

- $G = SO(3,1)\,, \qquad H = \mathbb{R}^4\,, \qquad L = \mathbb{R}^8$ (Grassmannians) ,
- $M_{ab} \triangleright P_c = [M_{ab}, P_c]\,, \qquad M_{ab} \triangleright P_\alpha = \frac{1}{2}(\sigma_{ab})^\beta{}_\alpha P_\beta\,, \qquad M_{ab} \triangleright P^\alpha = -\frac{1}{2}(\sigma_{ab})^\alpha{}_\beta P^\beta\,,$
- $\partial(P_a) = 0\,, \qquad \delta(T_A) = 0\,, \qquad \{P_a\,, P_b\} = 0\,.$

↪ 3-connection $(\alpha\,,\beta\,,\gamma)$: $\boxed{\alpha = \omega^{ab} M_{ab}\,, \qquad \beta = \beta^a P_a\,, \qquad \gamma = \gamma^\alpha P_\alpha + \bar{\gamma}_\alpha P^\alpha}$.

↪ 3-curvature $(\mathcal{F}\,,\mathcal{G}\,,\mathcal{H})$:

$$\boxed{\begin{aligned} \mathcal{F} = R^{ab} M_{ab}\,, \qquad \mathcal{G} = \nabla\beta^a P_a\,, \\ \mathcal{H} = \left(\mathrm{d}\gamma^\alpha + \frac{1}{2}\omega^{ab}(\sigma_{ab})^\alpha{}_\beta \gamma^\beta\right) P_\alpha + \left(\mathrm{d}\bar{\gamma}_\alpha - \frac{1}{2}\omega^{ab}\bar{\gamma}_\beta (\sigma_{ab})^\beta{}_\alpha\right) P^\alpha \\ \equiv (\vec{\nabla}\gamma)^\alpha P_\alpha + (\bar{\gamma}\overleftarrow{\nabla})_\alpha P^\alpha\,. \end{aligned}}$$

↪ Topological action:

$$\boxed{S_{3BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla\beta^a + (\bar{\gamma}\overleftarrow{\nabla})_\alpha \psi^\alpha + \bar{\psi}_\alpha (\vec{\nabla}\gamma)^\alpha\,.}$$

# 3-GAUGE THEORY ON TRIANGULATION

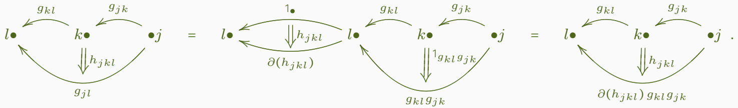
↪ Classical equations of motion impose the condition that the gauge connection is flat – that **every null-homotopic curve corresponds to the identity of the gauge group**.
↪ Within higher gauge theories, this condition is generalized by requiring that **the surface holonomy of the boundary 2-sphere of every 3-ball be trivial**.

↪ In the context of 3-gauge theory, the first condition remains unchanged, the second condition is generalized, while it is necessary to add the **condition of flatness of the boundary volume of the 4-simplex**.

**Lemma 1**

Zireli, Pfajfer, and Popesku arXiv: 0708.3051.

Consider the triangle $(jk\ell)$. The edges $(jk)$ are labeled by group elements $g_{jk} \in G$ and the triangles $(jk\ell)$ by elements $h_{jk\ell} \in H$.



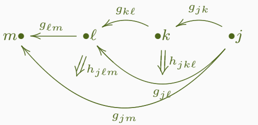

The curvature $\gamma_1 = g_{k\ell}g_{jk}$ is the source, and the curvature $\gamma_2 = g_{j\ell}$ is the target of the surface 2-morphism $\Sigma : \gamma_1 \to \gamma_2$, labeled by the group element $h_{jk\ell}$,

$$g_{j\ell} = \partial(h_{jk\ell})g_{k\ell}g_{jk}\,.$$

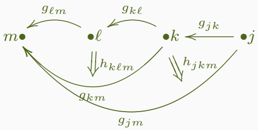
**Lemma 2**

TR and M. Vojinović, arXiv: 2201.02572.

Consider the tetrahedron $(jk\ell m)$. The tetrahedra $(jk\ell m)$ are labeled by group elements $l_{jk\ell m} \in L$.




$= (g_{\ell m} g_{j\ell}, h_{j\ell m}) \#_2 \big(g_{\ell m} \#_1 (g_{k\ell} g_{jk}, h_{jk\ell})\big) = \big(g_{\ell m} g_{k\ell} g_{jk}, h_{j\ell m}(g_{\ell m} \triangleright h_{jk\ell})\big)\,.$




$= (g_{km} g_{jk}, h_{jkm}) \#_2 \big((g_{\ell m} g_{k\ell}, h_{k\ell m}) \#_1 g_{jk}\big) = (g_{\ell m} g_{k\ell} g_{jk}, h_{jkm} h_{k\ell m})\,.$

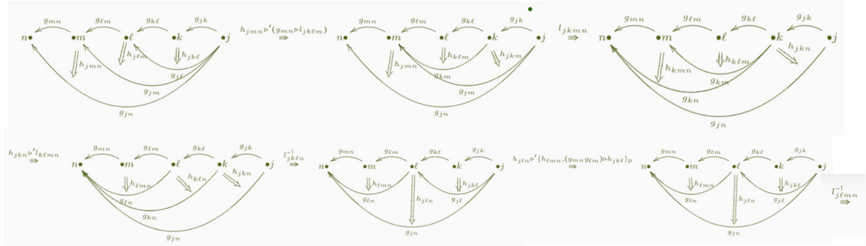
The mapping of the surface $\Sigma_1 : g_{\ell m} g_{k\ell} g_{jk} \to g_{jm}$ to the surface $\Sigma_2 : g_{\ell m} g_{k\ell} g_{jk} \to g_{jm}$ is determined by the element $l_{jk\ell m}$:

$$h_{jkm} h_{k\ell m} = \delta(l_{jk\ell m}) h_{j\ell m}(g_{\ell m} \triangleright h_{jk\ell})\,.$$

## Lemma 3

TR and M. Vojinović, arXiv: 2201.02572.

We consider a 4-simplex, $(jk\ell mn)$. We cut the 4-simplex volume along the surface $h_{jmn}g_{mn} \triangleright (h_{j\ell m}g_{\ell m} \triangleright h_{jk\ell})$.




This brings us back to the initial surface!

The obtained 3-morphism is the identity 3-morphism with source and target $\Sigma_1 = \Sigma_2 = h_{jmn}g_{mn} \triangleright (h_{j\ell m}g_{\ell m} \triangleright h_{jk\ell})$,

$$\boxed{l^{-1}_{j\ell mn}\, h_{j\ell n} \triangleright' \{h_{\ell mn}, (g_{mn}g_{\ell m}) \triangleright h_{jk\ell}\}_{\mathrm{p}}\, l^{-1}_{jk\ell n}(h_{jkn} \triangleright' l_{k\ell mn}) l_{jkmn} h_{jmn} \triangleright' (g_{mn} \triangleright l_{jk\ell m}) = e\,.}$$