

# Quantum mechanics for nonassociative quantum algebras

“quantum & fuzzy” Maja Fest

Peter Schupp

based on 2311.03647 [quant-ph] with Richard Szabo

**C>ONSTRUCTOR**  
UNIVERSITY

Belgrade 5 April 2024



# Examples of nonassociative (quantum) algebras

magnetic sources (monopoles)

interaction via deformation (of CR)

$$[P_i, P_j] = \underline{it\hbar F_{ij}}, \quad [x^i, p_j] = i\hbar \delta_j^i, \quad [x^i, x^j] = 0$$

undeformed Hamiltonian  $H = p^2/2m$

$$\leadsto \text{Lorentz-Heisenberg eom} \quad \dot{p}_i = \frac{i}{\hbar} [H, p_i] = \dot{x}^j F_{ij}$$

$$\leadsto \text{Jac}(p_i, p_j, p_k) dx^i dx^j dx^k = dF = *j_{\text{mag.}}$$

$\neq$  for magnetic sources!  $\leadsto$  NA!

weakened non-associativity : (popular in literature)

alternative alg.:  $x(xy) = (xx)y$   
e.g. octonions  $(yx)x = y(xx)$

$\leadsto$  associator  $[x, y, z] = (xy)z - x(yz)$  alternating

flexible alg.:  $x(yx) = (xy)x$   
e.g. Jordan alg.

power associative:  $(xx)x = x(xx) = x^3$  etc.

$\rightarrow$  we shall not assume any of these!



Lie algebras  $A = \mathbb{K} \oplus \mathfrak{g}$  with product  $[ab] = a \cdot b - b \cdot a$

$$[[a b] c] = [a [b c]] + [[a c] b] \neq [a [b c]] \quad \text{NA!}$$

→ associative universal enveloping algebras

Jordan algebras  $ab := \frac{1}{2}(a \cdot b + b \cdot a)$

$$\wedge a(bc) - (ab)c = \frac{1}{2} [[a; c]; b] \quad \text{non-associative, flexible}$$

normed division algebras

$\mathbb{R}, \mathbb{C}$   
commutative

$\mathbb{H}$  quaternions  
non-commutative  
associative

$\mathbb{O}$  octonions  
non-associative,  
alternative

# Dirac - Clifford algebra (Okubo 1986)

generators:  $\underbrace{1, \Gamma_5, [\gamma_\mu, \gamma_\nu]}_{\text{bosonic}}, \underbrace{\gamma_\mu, \Gamma_5 \gamma_\mu}_{\text{fermionic}} \quad (\mu = 1, 2, \dots, N)$   
generate  $SO(N+2)$

CRs as if  $\Gamma_5 = \gamma_5 = i\gamma_1\gamma_2\gamma_3\gamma_4$ , e.g.:  $\gamma_\mu\gamma_\nu = \mathbb{1}g_{\mu\nu} + \frac{1}{2}[\gamma_\mu, \gamma_\nu]$

for  $N \neq 4$  (dim. reg.):

nonassociative, not flexible, not power associative

trace:  $\text{Tr}(1) = N$ ,  $\text{Tr}(x) = 0$  for  $x \neq 1$

$\text{Tr}(xy) = \text{Tr}(yx)$ ,  $\text{Tr}(x(yz)) = \text{Tr}((xy)z)$   
2-cyclic                      3-cyclic

# non-geometric string back grounds

→ Poisson sigma model with 3-form R-flux:

$$S = \int_{\Sigma} \eta_{, \alpha} dx^{\alpha} + \frac{1}{2} \Theta^{\alpha\beta\gamma}(x) \eta_{, \alpha} \eta_{, \beta} \eta_{, \gamma}, \quad R = [\Theta, \Theta]_S$$

$$\Theta = \begin{pmatrix} R^{ijk} p_k & \delta^i_j \\ -\delta^i_j & 0 \end{pmatrix} \xrightarrow{\text{quant.}} \star \text{ non-associative!}$$

$$f \star g = \cdot \exp\left(\frac{i\hbar}{2} (R^{ijk} p_k \partial_i \otimes \partial_j + \partial_i \otimes \tilde{\partial}^i - \tilde{\partial}^i \otimes \partial_i)\right)$$

2-cyclic  $\int f \star g = \int g \star f$

3-cyclic  $\int f \star (g \star h) = \int (f \star g) \star h$

Mylonas, PS, Szabo 1402.7306


not Jordan,  
not power-assoc

A. Held, PS  
2014

# nonassociative QM challenge

want: observables, states, probability interpretation  
( $\downarrow$  need notions of reality & positivity)

problem:

QM: operators on Hilbert space  
 $\downarrow$  automatically associative! 

Solution:

distinguish  $\rightarrow$  operator algebra ( $\Rightarrow$  Univ. env. alg): associative  
 $\rightarrow$  quantum algebra: nonassociative



# Universal enveloping algebra

$A$ : unital alg. over  $k$  with unit 1

$$ab \neq ba$$

NC

$$a(bc) \neq (ab)c$$

NA

quantum algebra

$A^{\text{op}}$ : opposite algebra,  $\mathcal{U}(A) = \widehat{F}(A \oplus A^{\text{op}})$   
free assoc. alg. over  $k$

embedding  $\hat{\cdot}: A \oplus A^{\text{op}} \rightarrow \mathcal{U}(A)$ ,  $(a, b') \mapsto \hat{a} + \hat{b}'$

left action  $\triangleright: \mathcal{U}(A) \times A \rightarrow A$ :

$$\hat{a} \triangleright x = ax, \quad \hat{a}' \triangleright x = xa \quad (x \in A)$$

$\rightarrow$  multiplication algebra of  $A$

composition product  $\circ : \mathcal{U}(A) \times \mathcal{U}(A) \rightarrow \mathcal{U}(A)$

$$(\hat{a}_1 \circ \hat{a}_2) \triangleright x = \hat{a}_1 \triangleright (\hat{a}_2 \triangleright x) = a_1(a_2 x)$$

$$(\hat{b}'_1 \circ \hat{b}'_2) \triangleright x = \hat{b}'_1 \triangleright (\hat{b}'_2 \triangleright x) = (x b_2) b_1$$

$$(\hat{a}_1 \circ \hat{b}'_2)' = (\hat{b}'_2 \circ \hat{a}'_1) \triangleright x = b_2(x a_1) \text{ etc.}$$

$\circ$  is  $\circ$ -antihomomorphism

$$P: \mathcal{T}_0(A \oplus A^{op}) \rightarrow \mathcal{U}(A), \quad \mathcal{U}(A) \simeq \mathcal{T}_0(A \oplus A^{op}) / \ker(P)$$

relations in  $A$

we shall assume : relates left & right actions

•  $A$  is a unital  $\ast$ -algebra

• field  $\mathbb{K} = \mathbb{C}$

} as in traditional QM  
(but nonassociative)



# States and GNS construction

state:  $\mathbb{C}$ -linear functional  $\omega: \mathcal{U}(A) \rightarrow \mathbb{C}$

positive:  $\omega(A^* \circ A) \geq 0$ ,  $A \in \mathcal{U}(A)$

normalized:  $\omega(1) = 1$

Lemma (Cauchy-Schwarz)

$(A, B) \mapsto \omega(A^* \circ B)$  is semi-def. sesquilinear

$$\omega(A^* \circ B) = \overline{\omega(B^* \circ A)}$$

$$|\omega(A^* \circ B)|^2 \leq \omega(A^* \circ A) \omega(B^* \circ B)$$

proof  
hint

$$0 \leq \omega((z_1 A + z_2 B)^* \circ (z_1 A + z_2 B))$$

$$= (\bar{z}_1, \bar{z}_2) \cdot \underbrace{\begin{pmatrix} \omega(A^* \circ A) & \omega(A^* \circ B) \\ \omega(B^* \circ A) & \omega(B^* \circ B) \end{pmatrix}} \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$\Delta$  positive-semidef matrix

$\Rightarrow$  hermitean,  $\det(\dots) \geq 0 \dots \square$

↓ Gel'fand ideal of  $\omega$

$$\mathcal{I}_\omega := \{A \in \mathcal{U}(A) \mid \omega(A^* \circ A) = 0\}$$

is left ideal of  $\mathcal{U}(A)$  (proof via C.-S.)

quotient space  $\leadsto$  pre Hilbert space

$$\mathcal{H}_\omega := \mathcal{U}(A) / \mathcal{I}_\omega$$

with equivalence classes  $\Psi_A = \{\tilde{A} \in \mathcal{U} \mid \tilde{A} - A \in \mathcal{I}_\omega\}$

inner product

$$(\Psi_A, \Psi_B)_\omega = \omega(A^* \circ B)$$

lin. representation  $\pi_\omega : \mathcal{U}(A) \rightarrow \text{End}(\mathcal{H}_\omega)$

$$\pi_\omega(A) \psi_B := \psi_{A \circ B}$$

\*-rep.:  $\pi_\omega(A \circ B) = \pi_\omega(A) \pi_\omega(B)$

$$\pi_\omega(A^*) = \pi_\omega(A)^\dagger$$

cyclic with vacuum vector  $\psi_1$

$$\psi_A = \psi_{A \circ 1} = \pi_\omega(A) \psi_1$$

$$\langle A \rangle_\omega := \omega(A) = (\psi_1, \pi_\omega(A) \psi_1)_\omega$$

state  $\omega_\psi(A) := (\psi, \pi_\omega(A) \psi)_\omega \leftarrow \text{ref. state}$

# Observables

$$O \in \mathcal{U}(A), \quad O^* = O \text{ (real)}$$

$\hookrightarrow$  real expect. values :  $\omega(O) = \omega(O^*) = \overline{\omega(O)} \in \mathbb{R}$

uncertainty  $\Delta_\omega O := \sqrt{\omega((O - \langle O \rangle) \cdot (O - \langle O \rangle))}$

$$\Rightarrow \Delta_\omega O_1 \cdot \Delta_\omega O_2 \geq \frac{1}{2} |\langle [O_1, O_2] \rangle_\omega|$$

proof via Cauchy-Schwartz

## eigen states

$$\omega \text{ s.t. } \omega(B \circ A) = \lambda \omega(B), \quad \forall B \in \mathcal{U}(A)$$

↑  
eigenvalue

note: observables have real eigenvalues

$$\Leftrightarrow \omega((A - \lambda)^* \circ (A - \lambda)) = 0 \quad (\text{use C.-S.})$$

$$\Leftrightarrow \Pi_{\omega}(A) \psi_1 = \lambda \psi_1 \quad (\Leftrightarrow \psi_A = \lambda \psi_1)$$

eigenstates have zero uncertainty

( $\omega$  eigenstate of  $O \leadsto \Delta_{\omega} O = 0$ .)

# States from traces

Zooming in to non associativity

note:  
↙ not  $\mathcal{U}(A)$ !

trace  $\mathbb{C}$ -linear functional  $\tau: A \rightarrow \mathbb{C}$

• positive  $\tau(a^*a) \geq 0$

• normalized  $\tau(1) = 1$

• 2-cyclic  $\tau(ab) = \tau(ba)$

• 3-cyclic  $\tau(a(bc)) = \tau(c(ab))$

$\Rightarrow \tau(a(bc)) = \tau((ab)c)$

}  $a, b, c \in A$



$\tau$  is a state directly on quantum algebra  $A$ !

$\searrow$  tracial GNS construction

(gives smaller Hilbert space, "more nonassociative")

zero-norm vectors:  $\mathcal{J}_\tau = \{\psi \in A \mid \tau(\psi^*\psi) = 0\}$

$\psi \in \mathcal{J}_\tau \Rightarrow A \triangleright \psi \in \mathcal{J}_\tau$  (left ideal)

proof:  $\tau((A \triangleright \psi)^*(A \triangleright \psi)) = \tau((A^{op} \triangleright \psi^*)(A \triangleright \psi))$  use 3-cyclicity  $\searrow$

$$= \tau(\psi^*(A^{op} \triangleright (A \triangleright \psi))) = \tau(\psi^*((A^{op} \circ A) \triangleright \psi))$$

$$\leq \underbrace{\tau(\psi^*\psi)}_0 \tau(((A^{op} \circ A) \triangleright \psi)^*((A^{op} \circ A) \triangleright \psi))$$

C.S.

further states:  $\mu(a) := \tau(\psi^*(a\psi))$   
 $= \tau((a\psi)\psi^\#) = \tau(a \underbrace{(\psi\psi^\#)}_{S_\psi})$

more generally:  $\mu(a) = \sum_i p_i \tau(\psi_i^*(a\psi_i))$

with  $p_i \geq 0$ ,  $\sum p_i = 1$ ,  $\tau(\psi_i^* \psi_i) = 1$

positivity?

$\mu(\psi^\#((a^*a)\psi))$  is not positive  $\wedge$  no state on  $\mathcal{A}$

$\hookrightarrow$  need to go to  $\mathcal{U}(A) \dots$

$$\tau((a\psi)^*(a\psi)) \geq 0$$

$$= \tau(\psi^* a^* (a\psi)) = \tau(\psi^* ((\hat{a}^* \circ \hat{a}) \triangleright \psi))$$

3-cycl.

$\hookrightarrow$  notions of positivity, probability need  $\mathcal{U}(A)$

tracial state :  $\omega_{\tau}(A) = \tau(A \triangleright 1) =: \tau(A)$

→ state on both  $A$  and  $\mathcal{U}(A)$

proof uses 3-cyclicity

→ but not a trace on  $\mathcal{U}(A)$ :  $\tau(A \circ B) \neq \tau(B \circ A)$  !

↳ nonassociative effect.

further states on  $\mathcal{U}(A)$ :

$$\omega_{\psi}(A) = \sum p_i \tau(\psi_i^*(A \triangleright \psi_i)) \quad , \quad \sum p_i = 1, p_i \geq 0$$

we will suppress  $\sum p_i \dots$  in the following

# Dynamics

completely positive map:

map  $\phi$  from states to states that extends to  $\phi \otimes \text{id}$   
positive map CP

here:  $\phi: \omega \mapsto \tilde{\omega}$  CPT map

~ "trace-preserving"

$$\tilde{\omega}(O) = \sum_{k=1}^N \omega(A_k^\dagger \circ O \circ A_k), \quad \sum_{k=1}^N A_k^\dagger \circ A_k = 1$$

↑  
"Kraus operators"

Heisenberg pic.:  $\tilde{O} = \sum A_k^\dagger \circ O \circ A_k, \quad \tilde{1} = \sum A_k^\dagger \circ A_k = 1$

## Unitary time evolution

$$N=1, \text{ "Krauss op" } U_t = \exp_0\left(-\frac{i}{\hbar} t H\right), \quad H = H^*$$

Schrödinger-Liouville eqn:

$$\frac{d\omega_t(O)}{dt} = \frac{i}{\hbar} \omega_t([H, O])$$

$$\text{for } \psi : i\hbar \frac{d\psi}{dt} = H \psi$$

Heisenberg eqn:

$$\frac{dO}{dt} = \frac{i}{\hbar} [H, O]$$

examples:

## Jordan algebras

$(A, \cdot)$  NC, associative, unital  $\ast$ -algebra

commutative Jordan product:  $ab := \frac{1}{2}(a \cdot b + b \cdot a)$

original multiplication

non associative:  $a(bc) - (ab)c = \frac{1}{4} \llbracket a; c \rrbracket; b \rrbracket \neq 0$  i.g.

consider  $A = \overline{M_n(\mathbb{C})}$  with  $1 =$  unit matrix,  $\ast = +$

Prop.:  $\mathcal{U}(M_n(\mathbb{C})) = M_{n^2}(\mathbb{C})$



2-cyclic & 3-cyclic trace = normalized matrix trace

$$\tau(a) := \frac{1}{n} \text{Tr}(a)$$

unique square root

positive elements w.r.t.  $\tau$

$$a^* a = \frac{1}{2} (a^t \cdot a + a \cdot a^t) =: c = c^t = \sqrt{c} \cdot \sqrt{c} = \sqrt{c} \sqrt{c}$$

$\wedge$  density matrix

$$S_4 = \psi \psi^* =: \varphi \cdot \varphi, \quad \varphi = \sqrt{S_4}$$

... resolving ambiguities in previous treatment

$\wedge$  pure states = projectors  $\varphi^2 = \varphi = \varphi^*$

## tracial GNS representation

Gelfand's ideal trivial ( $\text{Tr}(a^+a) = 0 \Rightarrow a = 0$ )

$$\wedge \mathcal{H}_\tau = \mathbb{C}^{n^2}, \quad \pi_\tau(A)\psi = A \cdot \psi$$

## eigenstates

proposition:  $a \varphi = \lambda \varphi$ ,  $\varphi$  positive semi-def.

is solved by:  $\varphi = \sum p_i \phi_i \otimes \phi_i^+$ ,  $p_i \geq 0$

$a \cdot \phi_i = \lambda \phi_i$  (matrix - e. vec.)

## Dynamics examples

Jordan: any polynomial of observables is an observable

for  $y, z$  observables, consider:

$$H = \underbrace{\hbar\omega}_{\text{units of energy}} \underbrace{(z(yy) - (zy)y)}_{\text{would be zero for associative or even for alternative algebra!}}$$

$$= \frac{1}{4} \hbar\omega [[z; y], y] \neq 0$$

let  $(x, y, z) = (\sigma_1, \sigma_2, \sigma_3)$   $2 \times 2$  Pauli matrices

$$\underbrace{[\sigma_i; \sigma_j]}_{\text{su(2) Lie alg}} = 2i \epsilon_{ijk} \sigma_k, \quad \underbrace{\{\sigma_i; \sigma_j\}}_{\text{Cl(3) Clifford alg}} = 2\delta_{ij} 1$$

Jordan product:  $\sigma_i \sigma_j = \delta_{ij} 1$  {blind to  $SU(2)$   
sees only  $Cl(3)$ !}

$\Downarrow$   $H = \hbar \omega \hat{z}$  acts on  $\psi = a1 + b x + c y + d z$

$H \psi = \hbar \omega z \psi = \hbar \omega (a z + d 1) = i \hbar \dot{\psi}$

$\Downarrow$   $a(t) = \alpha \cos(\omega t + \phi), b(t) = \beta, c(t) = \gamma, d(t) = -i \alpha \sin(\omega t + \phi)$

# Octonions $A = \mathbb{O}$

$$e_0 = 1, \quad e_i e_j = -\delta_{ij} e_0 + \eta_{ijk} e_k \quad i, j, k \in \{1, \dots, 7\}$$

$\eta_{ijk}$  completely skewsym

$$\eta_{123} = \eta_{145} = \eta_{176} = \eta_{246} = \eta_{257} = \eta_{347} = \eta_{365} = 1$$

\*-involution:  $e_0^* = e_0, \quad e_i^* = -e_i$

2-cyclic & 3-cyclic trace:  $\tau(e_0) = 1, \quad \tau(e_i) = 0$

## enveloping algebra

left action of  $\mathcal{O}$  on itself defines  $8 \times 8$  matrix rep.  $\Pi$

$$\mathbb{1} := \Pi(e_0), \quad E_i := \Pi(e_i)$$

The  $E_i$  act as signed permutations, e.g.:

$$E_1 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\text{and } E_2 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$(01)_- (23)_- (45)_- (76)_-$

and

$(02)_- (31)_- (46)_- (57)_-$

let  $\bar{E}_0 = E_0, \bar{E}_i = -E_i \quad \wedge \quad \bar{E}_\mu \cdot E_\nu + \bar{E}_\nu \cdot E_\mu = 2\delta_{\mu\nu} \mathbb{1}$

$T_\mu = \begin{pmatrix} 0 & E_\mu \\ \bar{E}_\mu & 0 \end{pmatrix}$  generate  $Cl(8)$

proposition :  $\mathcal{U}(\mathcal{D}) = M_8(\mathbb{C})$

tracial GNS  $\tau(e_0) = 1, \tau(e_i) = 0$

$\mathcal{J}_\tau = \{0\}$  trivial  $\wedge$  Hilbert space  $\mathcal{H}_\tau = \mathbb{C}^8$

$\pi_\tau(e_0) = \mathbb{1}_{8 \times 8}, \pi_\tau(e_i) = E_i$

$\tau(A) = \pi_\tau(A)_{00}$  ( $(0,0)$ -matrix entry)



## tracial uncertainties

consider  $A := ie_7 = A^*$ ,  $B = e_1 \circ e_2 \circ e_4 = B^*$

$$A \circ A = B \circ B = \mathbb{1}, \quad A \circ B = -B \circ A = :iE, \quad E = E^*$$
$$E \circ E = \mathbb{1}$$

$$[A, B] = 2iE$$

$$\tau(E) = 1 \Rightarrow \tau([A, B]) \neq 0$$

← impossible for  
associative algs.

$$\underbrace{\Delta_{\tau} A}_{=1} \underbrace{\Delta_{\tau} B}_{=1} \geq \frac{1}{2} |\langle [A, B] \rangle_{\tau}| = 1$$

← in finite dim.

∴  $\tau$  is minimum uncertainty state

# Exotic quantum algebras, dual operator algebras

left operator alg

$$\hat{a} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \hat{b} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

associative  
commutative  
unital  
\*-algebra

$$\hat{0} = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

quantum algebra

$$\begin{array}{l} aa = a, ab = b \\ ba = -b, bb = a \end{array}$$

$(bb)b = b \neq -b = b(bb)$   
 $ab = b \neq -b = ba$   
non associative  
non power-assoc.  
non commutative  
non unital  
non \*

right (dual) op. algebra

$$\check{a} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z, \check{b} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_x$$

associative  
non commutative  
unital:  $\check{a}^2 = \check{b}^2 = 1$   
\*-algebra

## Summary

- natural generalization of QM:  
can handle nonassociative quantum algebras  
equals traditional QM for associative  $q$ -alg.
- GNS-inspired representation of operator alg.  
on quantum alg.  $\rightarrow$  univ. enveloping alg.
- preferred 3-cyclic state on quantum algebra
- completely positive dynamics
- bonafide nonassociative effects

