# The BV Formulation of Fuzzy Field Theories

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# Homotopy Algebras and Quantum Field Theory

- Higher structures in algebra and geometry have been playing a prominent role in many recent developments in physics

- In particular, homotopical methods based on $L_\infty$-algebras and $A_\infty$-algebras have increased our understanding of algebraic and kinematic structures of correlation functions of quantum field theory

- Quantum Batalin-Vilkovisky (BV) formalism gives explicit homological constructions in purely algebraic fashion without resorting to canonical quantization or path integral techniques

- **In this talk:** Apply modern incarnation of BV quantization (à la Costello-Gwilliam) to **fuzzy field theories**, which can be quantized in a completely rigorous way avoiding functional analytic complications of continuum field theories

  — they are also interesting examples of noncommutative field theories

# Outline

- Quantum BV Field Theory and $L_\infty$-Algebras
- Homological Perturbation Theory
- Scalar Field Theory on the Fuzzy Sphere
- Outlook

with Hans Nguyen and Alexander Schenkel [arXiv: 2107.02532]

# Free BV Field Theory $(E, Q_0, \langle -, - \rangle)$

- Graded vector space

  $E = \cdots \oplus E^{-1} \oplus E^0 \oplus E^1 \oplus \cdots = \text{ghosts} \oplus \text{fields} \oplus \text{antifields}$

  $Q_0 : E \longrightarrow E$ differential of degree 1 ($Q_0^2 = 0$)

  $\langle -, - \rangle : E \otimes E \longrightarrow \mathbb{C}$ non-degenerate graded antisymmetric of degree $-1$ and $Q_0$-invariant ($-1$-shifted symplectic structure)

- Describes *derived* space of free fields

- Polynomial observables $\big(\mathrm{Sym}\, E^* \simeq \mathrm{Sym}\, E[1], Q_0, \{-, -\}\big)$:

  BV antibracket $\{\varphi, \psi\} = \langle \varphi, \psi \rangle\, \mathbb{1}$ for $\varphi, \psi \in E[1]$ defines a $P_0$-algebra:

$$-Q_0\{\varphi, \psi\} = \{Q_0\varphi, \psi\} + (-1)^{|\varphi|}\, \{\varphi, Q_0\psi\} \quad \text{compatibility}$$

$$\{\varphi, \psi\} = (-1)^{|\varphi|\,|\psi|}\, \{\psi, \varphi\} \quad \text{symmetric}$$

$$\{\varphi, \{\psi, \chi\}\} = \pm\{\psi, \{\chi, \varphi\}\} \pm \{\chi, \{\varphi, \psi\}\} \quad \text{Jacobi identity}$$

$$\{\varphi, \psi\, \chi\} = \{\varphi, \psi\}\, \chi \pm \psi\, \{\varphi, \chi\} \quad \text{Leibniz rule}$$

# $L_\infty$-Algebras

- Extend cochain complex $(E[-1], Q_0)$ by antisymmetric maps $\{\ell_n : E[-1]^{\otimes n} \longrightarrow E[-1]\}_{n\geq 2}$ to form an $L_\infty$-algebra:

$$Q_0\ell_2(v,w) = \ell_2(Q_0 v, w) \pm \ell_2(v, Q_0 w) \qquad \text{Leibniz rule}$$

$$\ell_2(v,\ell_2(w,u)) + \mathrm{cyclic} = (Q_0 \circ \ell_3 \pm \ell_3 \circ Q_0)(v,w,u) \quad \text{Jacobi up to homotopy}$$

  plus "higher homotopy Jacobi identities"

- *Cyclic* with respect to pairing $\langle -,- \rangle : E[-1] \otimes E[-1] \longrightarrow \mathbb{C}$:

$$\langle v_0, \ell_n(v_1, v_2, \dots, v_n)\rangle = \pm \langle v_n, \ell_n(v_0, v_1, \dots, v_{n-1})\rangle$$

- (Cyclic) $L_\infty$-algebras are homotopy coherent generalizations of (quadratic) Lie algebras

- Extended $L_\infty$-algebra on $(\mathrm{Sym}\, E[1]) \otimes E[-1]$:

$$\ell_n^{\rm ext}(a_1 \otimes v_1, \dots, a_n \otimes v_n) = \pm\, a_1 \cdots a_n \otimes \ell_n(v_1, \dots, v_n)$$

$$\langle a_1 \otimes v_1, a_2 \otimes v_2\rangle_{\rm ext} = \pm\, a_1\, a_2\, \langle v_1, v_2\rangle$$

# Interacting BV Field Theory

- Interactions $I \in (\mathrm{Sym}\, E[1])^0$ incorporated by choosing dual bases $\varepsilon_\alpha \in E[-1]$ , $\varrho^\alpha \in E[-1]^* \simeq E[2]$ and 'contracted coordinate functions'
  $\mathsf{a} \;=\; \varrho^\alpha \otimes \varepsilon_\alpha \;\in\; \big((\mathrm{Sym}\, E[1]) \otimes E[-1]\big)^1$

- Homotopy Maurer-Cartan Action:

$$\lambda\, I \;=\; \sum_{n\geq 2} \frac{\lambda^{n-1}}{(n+1)!}\, \langle \mathsf{a}, \ell_n^{\mathrm{ext}}(\mathsf{a}, \dots, \mathsf{a})\rangle_{\mathrm{ext}} \;\in\; (\mathrm{Sym}\, E[1])^0$$

$$S_{\mathrm{BV}} \;=\; \langle \mathsf{a}, Q_0(\mathsf{a})\rangle_{\mathrm{ext}} + \lambda\, I \;=\; \text{BV action}$$

- (Classical) Master Equation: $Q_0(\lambda\, I) + \frac{1}{2}\,\{\lambda\, I, \lambda\, I\} \;=\; 0$

- $Q_{\mathrm{int}}^2 \;=\; 0$ where $Q_{\mathrm{int}} \;=\; Q_0 + \{\lambda\, I, -\}$

- Defines $P_0$-algebra $\big(\mathrm{Sym}\, E[1], Q_{\mathrm{int}}, \{-,-\}\big)$ of observables for interacting BV field theory

# Quantum BV Field Theory

- BV Laplacian $\Delta_{\rm BV} : \mathrm{Sym}\, E[1] \longrightarrow (\mathrm{Sym}\, E[1])[1]$:

$$\Delta_{\rm BV}(\mathbb{1}) = 0 = \Delta_{\rm BV}(\varphi) \quad , \quad \Delta_{\rm BV}(\varphi\,\psi) = \{\varphi,\psi\}$$

$$\Delta_{\rm BV}(a\,b) = \Delta_{\rm BV}(a)\,b + (-1)^{|a|}\,a\,\Delta_{\rm BV}(b) + \{a,b\}$$

$$\Delta_{\rm BV}\big(\varphi_1\cdots\varphi_n\big) = \sum_{i<j} \pm\{\varphi_i,\varphi_j\}\ \varphi_1\cdots\varphi_{i-1}\,\widehat{\varphi}_i\,\varphi_{i+1}\cdots\varphi_{j-1}\,\widehat{\varphi}_j\,\varphi_{j+1}\cdots\varphi_n$$

  Implements Gaussian integration/Wick's Theorem

- Satisfies $Q_0\,\Delta_{\rm BV} + \Delta_{\rm BV}\,Q_0 = 0$ , $\Delta_{\rm BV}^2 = 0$ , $\Delta_{\rm BV}(\lambda\,I) = 0$

- $Q_{\rm BV}^2 = 0$ where $Q_{\rm BV} = Q_{\rm int} + \hbar\,\Delta_{\rm BV} = Q_0 + \{\lambda\,I, -\} + \hbar\,\Delta_{\rm BV}$

- Quantum observables $\big(\mathrm{Sym}\, E[1], Q_{\rm BV}\big)$ ($E_0$-algebra) for interacting BV field theory

# Homological Perturbation Theory

- Propagators determine strong deformation retracts of $E^* \simeq E[1]$:

$$\big(H^\bullet(E[1]),0\big) \underset{\pi}{\overset{\iota}{\rightleftarrows}} \big(E[1],Q_0\big) \circlearrowleft \gamma \qquad \pi\,\iota = \mathbb{1}\ ,\ \iota\,\pi - \mathbb{1} = Q_0\,\gamma + \gamma\,Q_0$$
$$\gamma^2 = 0\ ,\ \gamma\,\iota = 0\ ,\ \pi\,\gamma = 0$$

- Observables: $\big(\mathrm{Sym}\,H^\bullet(E[1]),0\big) \underset{\Pi}{\overset{\mathcal{I}}{\rightleftarrows}} \big(\mathrm{Sym}\,E[1],Q_0\big) \circlearrowleft \Gamma$

- Maps $\mathcal{I}$ and $\Pi$ extend $\iota$ and $\pi$ as commutative dg-algebra morphisms:

$$\mathcal{I}([\psi_1]\cdots[\psi_n]) = \iota[\psi_1]\cdots\iota[\psi_n] \quad , \quad \Pi(\varphi_1\cdots\varphi_n) = \pi(\varphi_1)\cdots\pi(\varphi_n)$$

- $(\iota\,\pi)^2 = \iota\,\pi : E[1] \longrightarrow H^\bullet(E[1])$ splits $E[1] = E[1]^\perp \oplus H^\bullet(E[1])$:

$$\mathrm{Sym}\,E[1] = \mathrm{Sym}\,E[1]^\perp \otimes \mathrm{Sym}\,H^\bullet(E[1])$$

- Put $\Gamma(\varphi_1^\perp\cdots\varphi_n^\perp\otimes[\psi]) = \dfrac{1}{n}\displaystyle\sum_{i=1}^n \pm\,\varphi_1^\perp\cdots\varphi_{i-1}^\perp\,\gamma(\varphi_i^\perp)\,\varphi_{i+1}^\perp\cdots\varphi_n^\perp\otimes[\psi]$

# Homological Perturbation Theory

- Homological Perturbation Lemma: With $\delta = \{\lambda\, I, -\} + \hbar\, \Delta_{\rm BV}$, there is a strong deformation retract

$$\big(\mathrm{Sym}\, H^\bullet(E[1]), \widetilde{\delta}\,\big) \underset{\widetilde{\Pi}}{\overset{\widetilde{\mathcal{I}}}{\rightleftarrows}} \big(\mathrm{Sym}\, E[1], Q_{\rm BV}\big) \circlearrowleft \widetilde{\Gamma}$$

where $\widetilde{\Pi} = \Pi\, (\mathbb{1} - \delta\,\Gamma)^{-1}\, \delta\,\Gamma = \Pi \circ \sum_{k=1}^{\infty} (\delta\,\Gamma)^k$

- $\langle \varphi_1 \cdots \varphi_n \rangle := \widetilde{\Pi}(\varphi_1 \cdots \varphi_n) \in \mathrm{Sym}\, H^\bullet(E[1])$ are $n$-point correlation functions on space of vacua $H^\bullet(E)$ of the field theory

- Evaluated in a particular vacuum this gives the usual numerical correlations of perturbative quantum field theory around this vacuum

## Scalar Field Theory on the Fuzzy Sphere

- Fuzzy sphere: Take the spin $\alpha = \frac{N-1}{2}$ irrep of $su(2)$, with generators

$$[X_i, X_j] = \mathrm{i}\, r_N\, \epsilon_{ijk}\, X_k \ , \ X_i\, X_i = \mathbb{1} \ , \ X_i^* = X_i$$

$$A = (\alpha) \otimes (\alpha)^* \simeq \mathrm{Mat}(N)$$

- Free BV field theory: $E = E^0 \oplus E^1$ with $E^0 = E^1 = A$

$$Q_0 = \Delta + m^2 \quad , \quad \Delta(a) = \frac{1}{r_N^2}\,[X_i,[X_i,a]] \ \text{(fuzzy Laplacian)}$$

$$\langle \varphi, \psi \rangle = (-1)^{|\varphi|}\, \frac{4\pi}{N}\, \mathrm{Tr}(\varphi\, \psi)$$

- Fuzzy spherical harmonics: $Y_j^J \in A$ $(0 \leq J \leq N \ , \ -J \leq j \leq J)$ satisfy

$$\Delta(Y_j^J) = J\,(J+1)\, Y_j^J \quad , \quad \frac{4\pi}{N}\, \mathrm{Tr}(Y_j^{J\,*}\, Y_{j'}^{J'}) = \delta_{JJ'}\, \delta_{jj'}$$

$$Y_i^I\, Y_j^J = \sum_{K,k} \pm \sqrt{(2I+1)\,(2J+1)\,(2K+1)} \begin{pmatrix} I & J & K \\ i & j & -k \end{pmatrix} \begin{Bmatrix} I & J & K \\ \alpha & \alpha & \alpha \end{Bmatrix} Y_k^K$$

# Scalar Field Theory on the Fuzzy Sphere

- $L_\infty$-algebra: For any $n \geq 2$, choose $\ell_n : E[-1]^{\otimes n} \longrightarrow E[-1]$ as

$$\ell_n(\varphi_1, \dots, \varphi_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \varphi_{\sigma(1)} \cdots \varphi_{\sigma(n)}$$

- Contracted coordinate functions:

$$\mathsf{a} = \sum_{J,j} Y_j^{J*} \otimes Y_j^J \in \big((\mathrm{Sym}\, E[1]) \otimes E[-1]\big)^1$$

- Interactions: $\lambda\, I = \dfrac{\lambda^{n-1}}{(n+1)!} \displaystyle\sum_{\{J_i, j_i\}} I^{J_0 \cdots J_n}_{j_0 \cdots j_n}\, Y^{J_0 *}_{j_0} \cdots Y^{J_n *}_{j_n} \in (\mathrm{Sym}\, E[1])^0$

$$I^{J_0 \cdots J_n}_{j_0 \cdots j_n} = \langle Y^{J_0}_{j_0}, \ell_n(Y^{J_1}_{j_1}, \dots, Y^{J_n}_{j_n}) \rangle \in \mathbb{C}$$

expressed in terms of Wigner $3j$ and $6j$ symbols of $su(2)$

E.g. $$I^{J_0 \cdots J_3}_{j_0 \cdots j_3} = \prod_{i=0}^{3} \sqrt{2J_i + 1} \sum_{J,j} (-1)^j \, (2J+1) \times \begin{pmatrix} J_0 & J_1 & J \\ j_0 & j_1 & j \end{pmatrix} \begin{pmatrix} J_2 & J_3 & J \\ j_2 & j_3 & -j \end{pmatrix} \begin{Bmatrix} J_0 & J_1 & J \\ \alpha & \alpha & \alpha \end{Bmatrix} \begin{Bmatrix} J_2 & J_3 & J \\ \alpha & \alpha & \alpha \end{Bmatrix}$$

# Massive Scalar Field Theory on the Fuzzy Sphere

- Deformation retract: $H^\bullet(E[1]) = 0$ for $m^2 > 0$:

$$(0,0) \underset{0}{\overset{0}{\rightleftarrows}} (E[1], Q_0) \circlearrowleft -G \qquad G = Q_0^{-1} = (\Delta + m^2)^{-1}$$

- Correlation functions: $(\mathrm{Sym}\, 0 = \mathbb{C}, 0) \underset{\widetilde{\Pi}}{\overset{\widetilde{\mathcal{I}}}{\rightleftarrows}} (\mathrm{Sym}\, E[1], Q_{\mathrm{BV}}) \circlearrowleft \widetilde{\Gamma}$

  Only $\Pi(\mathbb{1}) = 1$ is non-zero (because $\pi = 0$)

- **Example:** 2-point function at 1-loop in $\Phi^4$-theory ($n = 3$):

$$\langle \varphi_1\, \varphi_2 \rangle = \Pi\big(\delta\,\Gamma(\varphi_1\,\varphi_2) + (\delta\,\Gamma)^2(\varphi_1\,\varphi_2) + (\delta\,\Gamma)^3(\varphi_1\,\varphi_2)\big)$$

$$= -\hbar\,\langle \varphi_1, G(\varphi_2)\rangle$$

$$- \frac{\lambda^2\,\hbar^2}{2} \sum_{\{J_i, j_i\}} \frac{I^{J_1 J J J_2}_{j_1 j j j_2}}{J\,(J+1) + m^2}\, \langle Y^{J_1}_{j_1}{}^*, G(\varphi_1)\rangle\, \langle Y^{J_2}_{j_2}{}^*, G(\varphi_2)\rangle + O(\lambda^4)$$

  Agrees with planar and non-planar loop corrections in conventional perturbation theory (Chu, Madore & Steinacker '01)

# Massless Scalar Field Theory on the Fuzzy Sphere

- Deformation retract: $H^\bullet(E[1]) \; = \; \mathbb{C}[1] \oplus \mathbb{C}$ for $m^2 = 0$:

$$\big(\mathbb{C}[1] \oplus \mathbb{C}\,,\,0\big) \underset{\frac{1}{N}\mathrm{Tr}}{\overset{\eta}{\rightleftarrows}} \big(E[1], Q_0\big) \circlearrowleft^{-G_0} \qquad G_0 \; = \; G^\perp \, \big(\mathrm{id}_A - \eta\,\tfrac{1}{N}\mathrm{Tr}\big)$$

$A \; = \; A^\perp \oplus \ker(\Delta)$ with projector $\eta\,\frac{1}{N}\mathrm{Tr} : A \longrightarrow \ker(\Delta) = \mathbb{C}\,\mathbb{1}_N$ where $\eta : \mathbb{C} \longrightarrow A$ is the unit map

By the Rank-Nullity Theorem of Linear Algebra, $\Delta$ restricts to an invertible map $\Delta^\perp : A^\perp \longrightarrow A^\perp$; extend $G^\perp \; = \; (\Delta^\perp)^{-1}$ by 0 to all of $A$

- Correlation functions: $\Pi(\mathbb{1}) \; = \; 1\,,\; \Pi(\varphi_1 \cdots \varphi_n) \; = \; \frac{1}{N}\mathrm{Tr}(\varphi_1) \cdots \frac{1}{N}\mathrm{Tr}(\varphi_n)$ in $\mathrm{Sym}\,\mathbb{C} \; = \; \mathbb{C}$, where

$$\tfrac{1}{N}\mathrm{Tr}(\varphi_i) : \ker(\Delta) \longrightarrow \mathbb{C} \quad , \quad \underline{\Phi} \; = \; \Phi_0\,\mathbb{1}_N \longmapsto \tfrac{1}{N}\mathrm{Tr}(\varphi_i\,\underline{\Phi}) \; = \; \tfrac{1}{N}\mathrm{Tr}(\varphi_i)\,\Phi_0$$

is a linear function on the space of vacua $\ker(\Delta)$

Purely classical contributions are completely analogous to expanding field operator $\widehat{\Phi} + \underline{\Phi}$ around generic classical solution $\underline{\Phi}$ in traditional QFT

## Outlook

- Fuzzy sphere has well-known *su*(2)-equivariant 3D differential calculus on $A = (\alpha) \otimes (\alpha)^*$ given by Chevalley-Eilenberg dg-algebra of *su*(2)

  Enables BV formulation of fuzzy field theories with gauge symmetries (Chern-Simons, Yang-Mills, etc.)

- Braided generalization of BV formalism (based on braided $L_\infty$-algebras) enables quantization of fuzzy field theories with braided symmetries from a purely algebraic perspective

  E.g. braided scalar field theory on the fuzzy torus — extension to gauge theories (differential calculus)? (Nguyen, Schenkel & Sz '21)

- Extensions to continuum (braided) models are possible (but not rigorous)
  (Giotopoulos & Sz '21; Dimitrijević Ćirić, Konjik, Radovanović & Sz '23; . . . )